Übungen Mehrdimensionale Analysis am 20. 04. 2024

1.)

Bestimmen Sie die allgemeine Lösung der nachstehenden Differentialgleichung mit Hilfe der Variation der Parameter für die partikuläre Lösung.

$$y'' + 2y' + y = e^{-x}.\ln(x)$$

2.) Lösen Sie die nachstehende inhomogene Differentialgleichung 4. Ordnung.

$$y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x}$$
$$y(1) = 0$$
$$y'(1) = e^2 - 3$$

3.)

Lösen sie die nachstehende DGL 2. Ordnung.

$$y'' - 4y' + 5y = \frac{e^{2x}}{\sin(x)}$$

4.)

Lösen sie die nachstehende DGL 2. Ordnung.

$$y'' + 2y' + 2y = \frac{e^{-x}}{\cos^3(x)}$$

1.) $y'' + 2y' + y = e^{-x} \cdot \ln(x)$

a.) homogene

$y'' + 2y' + y = 0$

$y = e^{\lambda x}$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda_{1,2} = -1 \pm \sqrt{1-1}$

$\underline{\underline{\lambda_1 = -1}}$ $\underline{\underline{\lambda_2 = -1}}$ gleiche λ-Werte.

$y_1 = e^{-x}$ $y_2 = x \cdot e^{-x}$

$\underline{\underline{y_h = A e^{-x} + B x e^{-x}}}$

b.) partikuläre

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$$

$$W = e^{-2x} - xe^{-2x} + xe^{-2x} = \underline{\underline{e^{-2x}}}$$

$$y_p = u \cdot y_1 + v \cdot y_2$$

$$u = -\int \frac{r(x) \cdot y_2(x)}{W}\, dx = -\int \frac{e^{-x} \cdot \ln(x) \cdot x e^{-x}}{e^{-2x}}\, dx$$

$$u = -\int \underset{g'}{x} \cdot \underset{h}{\ln(x)}\, dx = -\frac{x^2}{2}\ln(x) + \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx$$

$$u = -\frac{x^2}{2}\ln(x) + \frac{1}{2}\int x\,dx = \underline{\underline{-\frac{x^2}{2}\ln(x) + \frac{x^2}{4}}}$$

$$v = \int \frac{r(x)\, y_1}{W}\, dx = \int \frac{e^{-x} \cdot \ln(x) \cdot e^{-x}}{e^{-2x}}\, dx$$

$$v = \int \underset{g'}{1} \cdot \underset{h}{\ln(x)}\, dx = x\ln(x) - \int x \cdot \frac{1}{x}\, dx$$

$$\underline{\underline{v = x\ln(x) - x}}$$

$$y_p = \left(-\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right) \cdot e^{-x} + \left(x\ln(x) - x\right) \cdot x e^{-x}$$

$$y_p = -\frac{x^2}{2}\ln(x)\, e^{-x} + \frac{x^2}{4} e^{-x} + x^2 \ln(x)\, e^{-x} - x^2 e^{-x}$$

$$\underline{\underline{y_{allg} = A e^{-x} + B x e^{-x} + y_p}}$$

2.)

$$y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x} \qquad y(1) = 0 \quad y'(1) = e^2 - 3$$

<u>1.) Homogen DGL.</u>

$$r^2 - 4r + 4 = 0$$

$$r_{1,2} = 2 \pm \sqrt{4-4}$$

$$r_1 = 2 \quad \Rightarrow \quad \underline{y_1 = e^{2x}}$$

$$r_2 = 2 \quad \Rightarrow \quad \underline{\underline{y_2 = x \cdot e^{2x}}}$$

$$y_p = u \cdot y_1 + v \cdot y_2$$

$$W = y_1 y_2' - y_1' y_2 = e^{2x} \cdot (e^{2x} + 2xe^{2x}) - 2e^{2x} \cdot xe^{2x} = e^{4x} + \cancel{2xe^{4x}} - \cancel{2xe^{4x}} = \underline{\underline{e^{4x}}}$$

$$u = -\int \frac{y_2 \cdot r}{W} \cdot dx = -\int \frac{xe^{2x} \cdot \left(6 + \frac{e^{2x}}{x}\right)}{e^{4x}} \, dx$$

$$u = -\int 6x \cdot e^{-2x} + 1 \, dx = -\left(6x \cdot \left(-\tfrac{1}{2}\right) e^{-2x} - \int 6 \cdot \left(-\tfrac{1}{2}\right) e^{-2x} \, dx + x\right)$$

$$g \qquad h'$$

$$-\left(-3xe^{-2x} + \int 3e^{-2x} \, dx + x\right) = -\left(-3xe^{-2x} - \tfrac{3}{2} e^{-2x} + x\right) = \underline{\underline{3xe^{-2x} + \tfrac{3}{2} e^{-2x} - x}}$$

$$v = \int \frac{y_1 \cdot r}{W} \cdot dx = \int \frac{e^{2x} \cdot \left(6 + \frac{e^{2x}}{x}\right)}{e^{4x}} dx = \int 6e^{-2x} + \frac{1}{x} \, dx =$$

$$= \underline{\underline{-3e^{-2x} + \ln(x)}}$$

$$y_p = uy_1 + vy_2 = \cancel{3x} + \tfrac{3}{2} - xe^{2x} - \cancel{3x} + \ln(x) \cdot x \cdot e^{2x} = \underline{\underline{\tfrac{3}{2} - xe^{2x} + x \cdot \ln(x) \cdot e^{2x}}}$$

$$\underline{\underline{y_{allg}}} = \underline{\underline{C_1 e^{2x} + C_2 xe^{2x} + \tfrac{3}{2} - xe^{2x} + x \ln(x) e^{2x}}}$$

$$y(1) = 0 = C_1 e^2 + C_2 e^2 + \frac{3}{2} - e^2$$

$$y'(x) = 2C_1 e^{2x} + C_2 e^{2x} + 2C_2 x e^{2x} \cancel{- e^{2x}} - 2x e^{2x} + \ln(x) e^{2x} \cancel{+ e^{2x}} + 2x \ln(x) e^{2x}$$

$$= 2C_1 e^{2x} + C_2 e^{2x} + 2C_2 x e^{2x} - 2x e^{2x} + \ln(x) e^{2x} + 2x \ln(x) e^{2x}$$

$$y'(1) = e^2 - 3 = 2C_1 e^2 + C_2 e^2 + 2C_2 e^2 - 2e^2$$

$$2C_1 e^2 + 3C_2 e^2 = 3e^2 - 3$$

$$C_1 e^2 + C_2 e^2 = e^2 - \frac{3}{2}$$

$$2C_1 + 3C_2 = 3 - 3e^{-2}$$

$$C_1 + C_2 = 1 - \frac{3}{2}e^{-2} \quad | \cdot 2 \quad \} -$$

$$C_2 = 3 \cancel{- 3e^{-2}} - 2 \cancel{+ 3e^{-2}}$$

$$C_2 = 1$$

$$C_1 = -\frac{3}{2}e^{-2}$$

$$y(x) = -\frac{3}{2}e^{-2} \cdot e^{2x} + \cancel{x \cdot e^{2x}} + \frac{3}{2} \cancel{- x e^{2x}} + x \cdot \ln(x) e^{2x}$$

$$\boxed{y(x) = -\frac{3}{2}e^{-2} \cdot e^{2x} + \frac{3}{2} + x \ln(x) e^{2x}}$$

3) $y'' - 4y' + 5y = \dfrac{e^{2x}}{\sin(x)}$

a) homogène

$$r^2 - 4r + 5 = 0$$

$$r_{1,2} = 2 \pm \sqrt{4-5}$$

$$r_1 = 2+i \qquad r_2 = 2-i$$

$$\underline{\underline{y_h = e^{2x}\left(A\cos(x) + B\sin(x)\right)}}$$

$$y_1 = e^{2x}\cos(x) \qquad y_2 = e^{2x}\sin(x)$$

$$W = \begin{vmatrix} e^{2x}\cos(x) & e^{2x}\sin(x) \\ 2e^{2x}\cos(x) - e^{2x}\sin(x & 2e^{2x}\sin(x) + e^{2x}\cos(x) \end{vmatrix}$$

$$W = \cancel{2e^{4x}\sin(x)\cos(x)} + e^{4x}\cos^2(x) - \cancel{2e^{2x}\sin(x)\cos(x)} + e^{4x}\sin^2(x)$$

$$\underline{\underline{W = e^{4x}}}$$

$$u = -\int \frac{r(x)\cdot y_2(x)}{W}\,dx = -\int \frac{e^{2x}\,e^{2x}\sin(x)}{\sin(x)\cdot e^{4x}}\,dx = \underline{\underline{-x}}$$

$$V = \int \frac{g(x) \cdot y_1(x)}{W} \, dx = \int \frac{e^{2x} \cdot e^{2x} \cdot \cos(x)}{\sin(x) \cdot e^{4x}} \, dx = \underline{\underline{\ln(\sin}}$$

$$\underline{\underline{y_p = -x \cdot e^{2x} \cdot \cos(x) + \ln(\sin(x))\, e^{2x} \sin(x)}}$$

$$y_{allg} = y_h + y_p$$

4.) $y'' + 2y' + 2y = \dfrac{e^{-x}}{\cos^3(x)}$

a.) homogene.

$$r^2 + 2r + 2 = 0$$

$$r_{1,2} = -1 \pm \sqrt{1-2}$$

$\underline{r_1 = -1 + i}$ $\qquad\qquad$ $\underline{r_2 = -1 - i}$

$$y_1 = e^{-x}\cos(x) \qquad\qquad y_2 = e^{-x}\sin(x)$$

$$\underline{\underline{y_h = C_1 e^{-x}\cos(x) + C_2 e^{-x}\sin(x)}}$$

b.) partikulär.

$$W = \begin{vmatrix} e^{-x}\cos(x) & e^{-x}\sin(x) \\ -e^{-x}\cos(x) - e^{-x}\sin(x) & -e^{-x}\sin(x) + e^{-x}\cos(x) \end{vmatrix}$$

$$= -e^{-2x}\cancel{\sin(x)\cos(x)} + e^{-2x}\cos^2(x) + e^{-2x}\cancel{\sin(x)\cos(x)} + e^{-2x}\sin^2(x)$$

$$\underline{\underline{W = e^{-2x}}}$$

$$y_p = u \cdot y_1 + v \cdot y_2$$

$$u = -\int \frac{r(x) \cdot y_2(x)}{W} \, dx$$

$$u = -\int \frac{e^{-x} \cdot e^{-x} \sin(x)}{\cos^3(x) \cdot e^{-2x}} \, dx = -\int \frac{\sin(x)}{\cos^3(x)} \, dx$$

$$u = -\int \tan(x) \cdot \frac{1}{\cos^2(x)} \, dx = \underline{\underline{-\frac{1}{2} \tan^2(x)}}$$

$$v = \int \frac{r(x) \, y_1}{W} \, dx = \int \frac{e^{-x} \cdot e^{-x} \cos(x)}{\cos^3(x) \; e^{-2x}} \, dx = \underline{\underline{\tan(x)}}$$

$$\underline{\underline{y_p = -\frac{1}{2} \tan^2(x) \, e^{-x} \cos(x) + \tan(x) \, e^{-x} \sin(x) = \frac{1}{2} \tan(x) \cdot \sin(x) \, e^{-x}}}$$

$$y_{allg} = y_h + y_p$$