Übungen Mehrdimensionale Analysis am 13. 04. 2024

1.)

Bestimmen sie die allgemeine Lösung der nachstehenden Differentialgleichung zweiter Ordnung die die gegebenen Randbedingungen.

$$y'' - 2y' + 2y = 0$$

$$y(0) = -3$$

$$y(\frac{\pi}{2}) = 0$$

2.)

Lösen Sie nachstehende Differentialgleichung.

$$y'' + 3y' + 2y = 2x^2$$

3.)

Lösen sie die nachstehende DGL 2. Ordnung.

$$y'' = \frac{1}{\cosh(y')} \qquad y(0) = 1 \quad y'(0) = 0$$

4.)

Lösen sie die nachstehende DGL 2. Ordnung.

$$\frac{y''}{y'} - \frac{y'}{y} = \ln(y)$$

1.) $y'' - 2y' + 2y = 0$

$y(0) = -3 \qquad y\left(\frac{\pi}{2}\right) = 0$

$y = e^{rx}$: Konstante Koeffizienten

$r^2 - 2r + 2 = 0$

$r_{1,2} = 1 \pm \sqrt{1-2} = 1 \pm \sqrt{-1}$

$r_1 = 1+i \qquad r_2 = 1-i$

$y = e^{x} \cdot (A\cos(x) + B\sin(x))$

$y(0) = -3 = A \qquad A = -3$

$y\left(\frac{\pi}{2}\right) = 0 = B \qquad B = 0$

$y = -3e^{x}\cos(x)$

Probe: $y = -3e^{x}\cos(x)$

$y' = -3e^{x}\cos(x) + 3e^{x}\sin(x)$

$y'' = -3\cancel{e^{x}\cos(x)} + 3\,2e^{x}\sin(x) + \cancel{3e^{x}\cos(x)}$

$\cancel{6e^{x}\sin(x)} + \cancel{6e^{x}\cos(x)} - \cancel{6e^{x}\sin(x)} - \cancel{6e^{x}\cos(x)} = 0$

$0 = 0$ ✓

2.) $y'' + 3y' + 2y = 2x^2$

a.) homogen.

$$y'' + 3y' + 2y = 0$$

$y = e^{\lambda x}$: $\lambda^2 + 3\lambda + 2 = 0$

$$\lambda_{1/2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}$$

$$\lambda_{1,2} = -\frac{3}{2} \pm \frac{1}{2}$$

$$\underline{\underline{\lambda_1 = -1}} \qquad \underline{\underline{\lambda_2 = -2}}$$

$$\underline{\underline{y_h(x) = Ae^{-x} + Be^{-2x}}}$$

b.) partikulär.

$$y_p = K_0 + K_1 x + K_2 x^2$$

$$y_p' = K_1 + 2K_2 x$$

$$y_p'' = 2K_2$$

$$2K_2 + 3K_1 + 6K_2 x + 2K_0 + 2K_1 x + 2K_2 x^2 = 2x^2$$

$2K_2 + 3K_1 + 2K_0 = 0 \qquad \Rightarrow K_0 = \frac{1}{2} \cdot (-2K_2 - 3K_1) = \frac{1}{2} \cdot (-2 + 9) = \underline{\underline{\frac{7}{2}}}$

$x \cdot (6K_2 + 2K_1) = 0 \qquad \Rightarrow \underline{\underline{K_1 = -3}}$

$x^2 \cdot (2K_2) = 2x^2 \qquad \Rightarrow \underline{\underline{K_2 = 1}}$

$$y_p = \frac{7}{2} - 3x + x^2$$

Probe für y_p:

$$y_p = \frac{7}{2} - 3x + x^2$$

$$y_p' = -3 + 2x$$

$$y_p'' = 2$$

$$2 + 3(-3+2x) + 2\left(\frac{7}{2} - 3x + x^2\right) = 2x^2$$

$$\cancel{2} - \cancel{9} + \cancel{6x} + \cancel{7} - \cancel{6x} + 2x^2 = 2x^2$$

$$y = Ae^{-x} + Be^{-2x} + \frac{7}{2} - 3x + x^2$$

3.) $y'' = \dfrac{1}{\cosh(y')}$

$y(0) = 1 \qquad y'(0) = 0$

$y' = u$

$y'' = u'$

$u' = \dfrac{1}{\cosh(u)}$

$\dfrac{du}{dx} = \dfrac{1}{\cosh(u)}$

$\cosh(u) \cdot du = dx$

$\sinh(u) = x + C_1$

$$\underline{\underline{u = \operatorname{arc\,sinh}(x + C_1)}}$$

$\dfrac{dy}{dx} = \operatorname{arcsinh}(x + C_1)$

$y = \int \underset{u'}{1} \cdot \underset{v}{\operatorname{arc\,sinh}(x + C_1)} \cdot dx$

$y = x \cdot \operatorname{arcsinh}(x + C_1) - \int x \cdot \dfrac{1}{\sqrt{1 + (x+c)^2}} \cdot dx$

$y = x \cdot \operatorname{arcsinh}(x + C_1) - \int \dfrac{x + c}{\sqrt{1 + (x+c)^2}} - \dfrac{c}{\sqrt{1+(x+c)^2}} \, dx$

$$y = x \cdot \operatorname{arcsinh}(x+C) - \sqrt{1+(x+C)^2} + C \cdot \operatorname{arcsinh}(x+C) + D$$

$$\underline{\underline{y(x) = (x+C) \cdot \operatorname{arcsinh}(x+C) - \sqrt{1+(x+C)^2} + D}}$$

$$y(0) = 1 = -\sqrt{1+C^2} + D$$

$$y'(0) = 0 = \operatorname{arcsinh}(0+C) \rightarrow \underline{\underline{C=0}} \rightarrow D = 2$$

$$\underline{\underline{y(x) = x \cdot \operatorname{arcsinh}(x) - \sqrt{1+x^2} + 2}}$$

<u>Probe:</u> $y' = \operatorname{arcsinh}(x) + x \cdot \cancel{\frac{1}{\sqrt{1+x^2}}} - \cancel{\frac{1 \cdot x}{\sqrt{1+x^2}}}$

$$y'' = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{1}{\sqrt{1+x^2}} = \frac{1}{\cosh(\operatorname{arcsinh}(x))} = \frac{1}{\sqrt{1+\sinh^2(\operatorname{arcsinh}(x))}} = \frac{1}{\sqrt{1+x^2}}$$

4.)

$$\frac{y''}{y'} - \frac{y'}{y} = \ln(y)$$

Fall b): $y' = u \quad y'' = u \cdot \frac{du}{dy}$

$$\frac{\cancel{u} \cdot \frac{du}{dy}}{\cancel{u}} - \frac{u}{y} = \ln(y)$$

$$\frac{du}{dy} - \frac{u}{y} = \ln(y)$$

a) homogen

$$\frac{du}{dy} = \frac{u}{y}$$

$$\frac{du}{u} = \frac{dy}{y}$$

$$\ln(u) = \ln(y) + \ln(c)$$

$$\underline{\underline{u_h = C \cdot y}}$$

b) partikulär.

$$u_p = C(y) \cdot y$$

$$u_p' = C'y + C$$

$$C'y + \cancel{C} - \cancel{C} = \ln(y)$$

$$C' = \frac{1}{y} \cdot \ln(y)$$

$$\underline{\underline{C = \frac{1}{2} \cdot \ln^2(y)}}$$

$$u_p = \frac{1}{2} \cdot y \cdot \ln^2(y)$$

$$\underline{\underline{u_{allg} = Cy + \frac{1}{2} y \cdot \ln^2(y)}} = y'$$

$$y' = Cy + \frac{y}{2}\ln^2(y)$$

$$\frac{dy}{Cy + \frac{y}{2}\ln^2(y)} = dx$$

$$\frac{dy}{Cy\cdot\left(1 + \frac{1}{2C}\cdot\ln^2(y)\right)} = dx$$

Subst: $\frac{1}{\sqrt{2C}}\cdot\ln(y) = z$

$$\frac{1}{\sqrt{2C}}\cdot\frac{1}{y}\cdot dy = dz$$

$$dy = \sqrt{2C}\cdot y\cdot dz$$

$$\frac{\sqrt{2C}\cdot\cancel{y}\cdot dz}{C\cdot\cancel{y}\cdot(1+z^2)} = dx$$

$$\frac{\sqrt{2}}{\sqrt{C}}\arctan(z) = x + D$$

$$\arctan(z) = \frac{\sqrt{C}}{\sqrt{2}}\cdot x + \frac{\sqrt{C}}{\sqrt{2}}\cdot D$$

$$z = \tan\left(\frac{\sqrt{C}}{\sqrt{2}}x + \frac{\sqrt{C}}{\sqrt{2}}\cdot D\right)$$

$$\frac{1}{\sqrt{2C}}\cdot\ln(y) = \tan(\text{—''—})$$

$$\ln(y) = \sqrt{2C}\cdot\tan(\,..\,\text{''—})$$

$$y = e^{\sqrt{2C}\cdot\tan\left(\frac{\sqrt{C}}{\sqrt{2}}x + \frac{\sqrt{C}}{\sqrt{2}}\cdot D\right)}$$

$$y = e^{\sqrt{2C}\cdot\tan\left(\frac{\sqrt{C}}{\sqrt{2}}x + \frac{\sqrt{C}}{\sqrt{2}}\cdot D\right)}$$

$$y = e^{A\cdot\tan\left(\frac{A}{2}\cdot x + B\right)}$$

Probe:

$$y = e^{A\cdot\tan\left(\frac{A}{2}x + B\right)}$$

$$y' = e^{A\tan\left(\frac{A}{2}x + B\right)} \cdot A \frac{1}{\cos^2\left(\frac{A}{2}x+B\right)} \cdot \frac{A}{2} = \underline{\underline{\frac{A^2}{2} \cdot \frac{e^{A\tan\left(\frac{A}{2}x+B\right)}}{\cos^2\left(\frac{A}{2}x+B\right)}}}$$

$$y'' = \frac{A^2}{2} \cdot \frac{e^{A\tan\left(\frac{A}{2}x+B\right)} \cdot A \cancel{\frac{1}{\cos^2\left(\frac{A}{2}x+B\right)}} \cdot \frac{A}{2} \cdot \cancel{\cos^2\left(\frac{A}{2}x+B\right)} - e^{A\tan\left(\frac{A}{2}x+B\right)} \cdot \cancel{2}\cos\left(\frac{A}{2}x+B\right) \cdot \left(-\sin\left(\frac{A}{2}x+B\right)\right) \cdot \frac{A}{\cancel{2}}}{\cos^4\left(\frac{A}{2}x+B\right)}$$

$$\underline{\underline{y'' = \frac{A^4}{4} \frac{e^{A\tan\left(\frac{A}{2}x+B\right)}}{\cos^4\left(\frac{A}{2}x+B\right)} + \frac{A^3}{2} e^{A\tan\left(\frac{A}{2}x+B\right)} \cdot \frac{1}{\cos^3\left(\frac{A}{2}x+B\right)} \cdot \sin\left(\frac{A}{2}x+B\right)}}$$

$$\frac{y''}{y'} - \frac{y'}{y} = \ln(y)$$

$$\cancel{\frac{A^2}{2} \frac{1}{\cos^2\left(\frac{A}{2}x+B\right)}} + A \cdot \frac{\sin\left(\frac{A}{2}x+B\right)}{\cos\left(\frac{A}{2}x+B\right)} - \cancel{\frac{A^2}{2} \frac{1}{\cos^2\left(\frac{A}{2}x+B\right)}} = A \cdot \frac{\sin\left(\frac{A}{2}x+B\right)}{\cos\left(\frac{A}{2}x+B\right)}$$

$0 = 0$