Übungen Mehrdimensionale Analysis am 23. 03. 2024

1.)

Lösen sie die folgende lineare Differentialgleichung 1. Ordnung.

$$y' - y.\cot(x) = 2x - x^2.\cot(x)$$

2.)

Lösen sie die folgende Bernoulli Differentialgleichung.

$$y' - x.y = x^3.y^2$$

3.)

Lösen sie die folgende Bernoulli Differentialgleichung.

$$y' + x.y = x.y^{-1}$$

4.)

Berechnen Sie die Lösung der nachstehenden Differentialgleichung

$$y' + \frac{1}{1+x} y = -(1+x)\, y^4$$

5.)

Lösen sie die nachstehende Differentialgleichung .

$$2x^2 y y' - y^2 = x^2 e^{x - \frac{1}{x}}$$

1.) $$y' - y \cdot \cot(x) = 2x - x^2 \cot(x).$$

a.) homogene

$$y' = y \cdot \cot(x)$$

$$\frac{dy}{y} = \frac{\cos(x)}{\sin(x)} \, dx$$

$$\ln(y) = \ln(\sin(x)) + \ln(C)$$

$$\underline{\underline{y_h = C \cdot \sin(x)}}$$

b) partikulär:

$$y_p = C(x) \cdot \sin(x)$$

$$y_p' = C' \sin(x) + C \cos(x)$$

$$C' \sin(x) + \cancel{C \cos(x)} - \cancel{C \sin(x) \cdot \frac{\cos(x)}{\sin(x)}} = 2x - x^2 \cdot \frac{\cos(x)}{\sin(x)}$$

$$C' = \frac{2x \sin(x) - x^2 \cos(x)}{\sin(x)^2} = \underbrace{\frac{2x}{\sin(x)}}_{\mathrm{I}} - \underbrace{\frac{x^2}{\sin^2(x)} \cos(x)}_{\mathrm{II}}$$

$$\mathrm{I}: \; 2 \int \underbrace{x}_{g'} \cdot \underbrace{\sin^{-1}(x)}_{h} \, dx = 2 \cdot \left(\frac{x^2}{2} \sin^{-1}(x) - \int \frac{x^2}{2} \cdot \frac{(-1)}{\sin^2(x)} \cos(x) \, dx \right.$$

$$= \frac{x^2}{\sin(x)} + \underbrace{\int x^2 \frac{\cos(x)}{\sin(x)} \, dx}_{\mathrm{II}} \;\rightarrow\; \underline{\underline{C = \frac{x^2}{\sin(x)}}}$$

$$y_p = x^2$$

$$\underline{\underline{y_{allg} = C \cdot \sin(x) + x^2}}$$

Probe: $y' = C\cos(x) + 2x$

$$\cancel{C\cos(x)} + \cancel{2x} - \cancel{C\cos(x)} - \underline{x^2 \cot(x)} = \cancel{2x} - \underline{x^2 \cot(x)}$$

$$\underline{\underline{0 = 0}} \quad \checkmark$$

2.) $y' - xy = x^3 y^2$ Bernoulli DGL

$u = y^{1-\alpha} = y^{1-2} = y^{-1}$

$y = u^{-1}$

$y' = -\frac{1}{u^2} \cdot u'$

$-\frac{1}{u^2} u' - x \frac{1}{u} = x^3 \frac{1}{u^2} \Big| \cdot (-u^2)$

$\underline{u' + x \cdot u = -x^3}$ "Lineare DGL"

a) homogen.

$u' = -x \cdot u$

$\frac{du}{u} = -x\,dx$

$\ln|u| = -\frac{x^2}{2} + \ln(c)$

$\underline{\underline{u_h = C \cdot e^{-\frac{x^2}{2}}}}$

b) partikulär $u_p = C(x)\, e^{-\frac{x^2}{2}}$, $u_p' = c' e^{-\frac{x^2}{2}} + C e^{-\frac{x^2}{2}} (-x)$

$c' e^{-\frac{x^2}{2}} \cancel{- C x e^{-\frac{x^2}{2}}} + \cancel{C x e^{-\frac{x^2}{2}}} = -x^3$

$C = -\int x^3 e^{+\frac{x^2}{2}} dx$

$$C = -\int x^3 e^{\frac{x^2}{2}} dx = -\int \underbrace{x^2}_{g} \cdot \underbrace{x \cdot e^{\frac{x^2}{2}}}_{h'} dx$$

$$C = -\left(x^2 e^{\frac{x^2}{2}} - \int 2x\, e^{\frac{x^2}{2}} dx \right)$$

$$\underline{\underline{C = -x^2 e^{\frac{x^2}{2}} + 2e^{\frac{x^2}{2}}}} \qquad \underline{\underline{u_p = -x^2 + 2}}$$

$$u_{allg} = Ce^{-\frac{x^2}{2}} - x^2 + 2 = \frac{1}{y}$$

$$\underline{y = \frac{1}{Ce^{-\frac{x^2}{2}} - x^2 + 2}}$$

3.)

$$y' + xy = \frac{x}{y} \qquad \alpha = -1$$

$$u = y^{1-\alpha} = y^2$$

$$u' = 2y \cdot y'$$

$$y' = \frac{u'}{2y}$$

$$\frac{u'}{2y} + xy = \frac{x}{y} \quad \Big| \cdot 2y$$

$$u' + 2xu = 2x$$

$$u' = 2x \cdot (1-u)$$

$$\frac{du}{1-u} = 2x\,dx$$

$$-\ln(1-u) = x^2 + \ln(c)$$

$$\ln(1-u) = -x^2 + \ln(c)$$

$$1-u = c \cdot e^{-x^2}$$

$$u = ce^{-x^2} + 1$$

$$y = \sqrt{Ce^{-x^2} + 1}$$

4.) $y' + \frac{1}{1+x} \cdot y = -(1+x) \cdot y^4$

Bernoulli -DGL:

$u = y^{1-\alpha} \qquad \alpha = 4$

$u = y^{-3}$

$u' = -3y^{-4} \cdot y' \qquad y' = -\frac{u' \cdot y^4}{3}$

$-\frac{u' \cdot y^4}{3} + \frac{1}{1+x} \cdot y = -(1+x) \cdot y^4 \,/ \cdot \left(\frac{3}{y^4}\right)$

$-u' + \frac{3}{1+x} \cdot y^{-3} = -3 \cdot (1+x)$

$$\underline{\underline{u' - \frac{3}{1+x} \cdot u = 3 \cdot (1+x)}}$$ "Lineare DGL"

a.) homogene

$u' - \frac{3}{1+x} \cdot u = 0$

$u' = \frac{3}{1+x} \cdot u$

$\frac{du}{u} = \frac{3}{1+x} \cdot dx$

$\ln(u) = 3 \cdot \ln(1+x) + \ln(c)$

$$\underline{\underline{u_h = c \cdot (1+x)^3}}$$

b) partikulär

$$u_p = C(x) \cdot (1+x)^3$$

$$u_p' = C' \cdot (1+x)^3 + C \cdot 3(1+x)^2$$

$$C' \cdot (1+x)^3 + \cancel{C \cdot 3 \cdot (1+x)^2} - \cancel{\frac{3}{1+x} \cdot C \cdot (1+x)^3} = 3 \cdot (1+x)$$

$$C' = \frac{3}{(1+x)^2}$$

$$C = 3 \cdot \int \frac{1}{(1+x)^2} \, dx$$

$$\underline{\underline{C = -\frac{3}{1+x}}}$$

$$u_p = -\frac{3}{1+x} \cdot (1+x)^3 = \underline{\underline{-3 \cdot (1+x)^2}}$$

$$u = C \cdot (1+x)^3 - 3 \cdot (1+x)^2$$

$$u = y^{-3}$$

$$\frac{1}{y^3} = u$$

$$y = \frac{1}{\sqrt[3]{u}}$$

$$\underline{\underline{y = \frac{1}{\sqrt[3]{C \cdot (1+x)^3 - 3 \cdot (1+x)^2}}}}$$

Probe: $y' + \frac{1}{1+x} \cdot y = -(1+x) \cdot y^4$

$$y = \left[C \cdot (1+x)^3 - 3 \cdot (1+x)^2\right]^{-1/3}$$

$$y' = -\frac{1}{3} \cdot \left(C \cdot (1+x)^3 - 3(1+x)^2\right)^{-4/3} \cdot \left(3C \cdot (1+x)^2 - 6 \cdot (1+x)\right)$$

$$\frac{-\left(\cancel{3}C \cdot (1+x)^2 - \overset{2}{\cancel{6}} \cdot (1+x)\right)}{\cancel{3} \cdot \left(C \cdot (1+x)^3 - 3 \cdot (1+x)^2\right)^{4/3}} + \frac{1}{1+x} \cdot \frac{1}{\left(C \cdot (1+x)^3 - 3 \cdot (1+x)^2\right)^{1/3}} = -(1+x) \cdot \frac{1}{\left(C \cdot (1+x)^3 - 3 \cdot (1+x)^2\right)}$$

$$\frac{-\cancel{C \cdot (1+x)^3} + 2 \cdot (1+x)^2 + 1 \cdot \left(\cancel{C \cdot (1+x)^3} - 3 \cdot (1+x)^2\right)}{\left(C \cdot (1+x)^3 - 3 \cdot (1+x)^2\right)^{4/3} \cdot (1+x)} = \frac{-(1+x) \cdot (1+x)}{(1+x) \cdot \left(C \cdot (1+x)^3 - 3 \cdot (1+x)^2\right)^4}$$

$$-(1+x)^2 = -(1+x)^2$$

$$0 = 0 \checkmark$$

Besser Weg:

$$y^{-3} = C \cdot (1+x)^3 - 3 \cdot (1+x)^2 \quad \Big| \frac{d}{dx}$$

$$-3 \cdot y^{-4} \cdot y' = C \cdot \left(3 \cdot (1+x)^2\right) - 6 \cdot (1+x)$$

aus Lösung: $\frac{y^{-3} + 3 \cdot (1+x)^2}{(1+x)^3} = C$

$$-\frac{3y'}{y^4} = \frac{3}{y^3(1+x)} + \frac{3 \cdot (1+x)^2}{(1+x)^3} \cdot 3 \cdot (1+x)^2 - 6 \cdot (1+x)$$

$$-\frac{\cancel{3}y'}{y^4} = \frac{\cancel{3}}{y^3 \cdot (1+x)} + \cancel{3} \cdot (1+x)$$

$$y' + \frac{1}{1+x} \cdot y = -(1+x)\, y^4$$

5.) $2x^2 y \cdot y' - y^2 = x^2 \cdot e^{x - \frac{1}{x}}$

$u = y^2$

$u' = 2yy'$

$x^2 \cdot u' - u = x^2 e^{x - \frac{1}{x}}$

$u' - \frac{1}{x^2} \cdot u = e^{x - 1/x}$

a.) Homogene

$u' - \frac{1}{x^2} \cdot u = 0$

$u' = \frac{1}{x^2} \cdot u$

$\frac{du}{u} = \frac{dx}{x^2}$

$\ln|u| = -\frac{1}{x} + \ln(c)$

$\underline{\underline{u_h = c \cdot e^{-\frac{1}{x}}}}$

b) partikuläre

$u_p = C(x) e^{-\frac{1}{x}}$

$u_p' = c' e^{-1/x} + c \cdot e^{-1/x} \cdot \left(\frac{1}{x^2}\right)$

$c' e^{-1/x} + \cancel{\frac{c}{x^2} e^{-1/x}} - \cancel{\frac{1}{x^2} \cdot c e^{-1/x}} = e^{x - \frac{1}{x}}$

$\underline{\underline{c' = e^x}}$ $\underline{\underline{c = e^x}}$

$$u_p = e^x \cdot e^{-1/x}$$

$$\underline{\underline{u_p = e^{x-\frac{1}{x}}}}$$

$$u = C e^{-1/x} + e^{x-\frac{1}{x}} = y^2$$

$$\underline{\underline{y = \sqrt{C e^{-1/x} + e^{x-1/x}}}}$$

<u>Probe:</u>

$$y' = \frac{1}{2} \cdot \frac{1}{\sqrt{}} \cdot \left(C \cdot e^{-1/x} \cdot \left(\frac{1}{x^2}\right) + e^{x-1/x} \cdot \left(1 + \frac{1}{x^2}\right)\right)$$

$$2x^2 y y' - y^2 = x^2 \cdot e^{x-\frac{1}{x}}$$

$$\cancel{2}x^2 \cdot \cancel{\sqrt{}} \cdot \frac{1}{\cancel{2}} \cdot \cancel{\frac{1}{\sqrt{}}} \left(C e^{-1/x} \cdot \frac{1}{x^2} + e^{x-1/x} \cdot \left(1 + \frac{1}{x^2}\right)\right)$$

$$- C e^{-1/x} - e^{x-1/x} = x^2 e^{x-1/x}$$

$$\cancel{C e^{-1/x}} + \underline{x^2 e^{x-1/x}} + \cancel{e^{x-1/x}} - \cancel{C e^{-1/x}} - \cancel{e^{x-1/x}} = \underline{x^2 e^{x-1/x}}$$

$$\underline{\underline{0 = 0}} \checkmark$$