Übungen Mathematik II am 04. 05. 2024

1.)

Lösen sie die nachstehende Differentialgleichung 2. Ordnung sowohl als Fall a.) als auch als Fall b.)

$$\frac{y''}{(1+y'^2)^{\frac{3}{2}}} = 1$$

2.)

Lösen sie das folgende schwierige Differentialgleichungssystem.

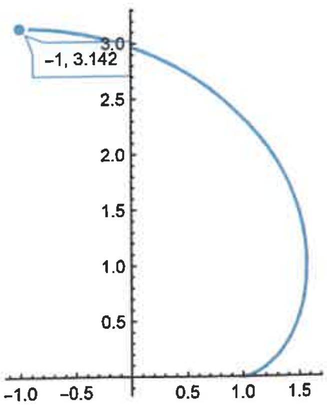
$$\dot{x} = -y.(x^2 + y^2)$$

$$\dot{y} = x.(x^2 + y^2)$$

3.)

Berechnen sie die Bogenlänge folgender Kurve von t = 0 bis t = π.

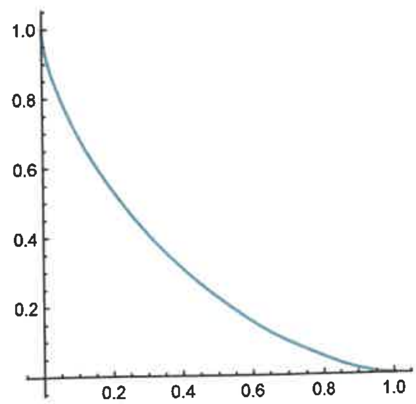
$$\vec{r} = \begin{pmatrix} \cos(t) + t.Sin(t) \\ \sin(t) - t.\cos(t) \end{pmatrix}$$



4.)

Berechnen sie die Bogenlänge folgender Kurve von t = 0 bis $t = \frac{\pi}{2}$.

$$\vec{r} = \begin{pmatrix} \cos(t)^3 \\ \sin(t)^3 \end{pmatrix}$$



1.)

$$\frac{y''}{(1+y'^2)^{3/2}} = 1$$

Fall a.) $\quad u = y' = \frac{dy}{dx}$

$y'' = u'$

$$\frac{u'}{(1+u^2)^{3/2}} = 1$$

$$\frac{du}{dx} = u' = (1+u^2)^{3/2}$$

$$\frac{du}{(1+u^2)^{3/2}} = dx$$

$$\int \frac{du}{(1+u^2)^{3/2}} = x + C$$

$$\cosh^2(t) - \sinh^2(t) = 1$$

$$\cosh^2(t) = 1 + \sinh^2(t)$$

Subst. $\quad u = \sinh(t)$

$\frac{du}{dt} = \cosh(t)$

$du = \cosh(t) \cdot dt$

$$\int \frac{\cosh(t) \cdot dt}{(1+\sinh^2(t))^{3/2}} = \int \frac{\cosh(t)\, dt}{\cosh^3(t)} = \int \frac{dt}{\cosh^2(t)} = \tanh(t)$$

$$= \frac{\sinh(t)}{\cosh(t)} = \frac{\sinh(t)}{\sqrt{1+\sinh^2(t)}} = \frac{u}{\sqrt{1+u^2}}$$

$$\frac{y'}{\sqrt{1+y'^2}} = (x+c)$$

$$y'^2 = (x+c)^2 \cdot (1+y'^2)$$

$$y'^2 = (x+c)^2 + (x+c)^2 \cdot y'^2$$

$$y'^2 \cdot (1-(x+c)^2) = (x+c)^2$$

$$y' = \frac{(x+c)}{\sqrt{1-(x+c)^2}}$$

$$\int dy = \int \frac{x+c}{\sqrt{1-(x+c)^2}} \cdot dx$$

$$y = -\sqrt{1-(x+c)^2} + D$$

$$(y+D)^2 = 1-(x+c)^2$$

$$\underline{\underline{(x+c)^2 + (y+D)^2 = 1}}$$

"Kreisgleichung"

1.)

$$\frac{y''}{(1+y'^2)^{3/2}} = 1$$

Fall b)

$$y' = u$$
$$y'' = u \cdot \frac{du}{dy}$$

$$\frac{u \cdot \frac{du}{dy}}{(1+u^2)^{3/2}} = 1$$

$$\frac{u \cdot du}{(1+u^2)^{3/2}} = dy$$

Subst. $u = \sinh(t)$

$du = \cosh(t)\, dt$

$$\frac{\sinh(t) \cdot \cosh(t) \cdot dt}{(1+\sinh^2(t))^{3/2}} = dy$$

$$\frac{\sinh(t) \cdot \cosh(t) \cdot dt}{\cosh^3(t)} = dy$$

$$\frac{\sinh(t)\, dt}{\cosh^2(t)} = dy$$

$$-\frac{1}{\cosh(t)} = y + C$$

$$-\frac{1}{\sqrt{1+\sinh^2(t)}} = y + C$$

$$-\frac{1}{\sqrt{1+u^2}} = (y+C)$$

$$-\sqrt{1+y'^2} = \frac{1}{y+c}$$

$$1+y'^2 = \frac{1}{(y+c)^2}$$

$$y'^2 = \frac{1}{(y+c)^2} - 1$$

$$y'^2 = \frac{1-(y+c)^2}{(y+c)^2}$$

$$y' = \frac{\sqrt{1-(y+c)^2}}{(y+c)}$$

$$\frac{y+c}{\sqrt{1-(y+c)^2}}\,dy = dx$$

$$-\sqrt{1-(y+c)^2} = x+D \quad \Big|^2$$

$$1-(y+c)^2 = (x+D)^2$$

$$\underline{\underline{(x+D)^2 + (y+c)^2 = 1}}$$

2.)

$$\dot{x} = -y\cdot(x^2+y^2)\quad\cdot x$$
$$\dot{y} = x\cdot(x^2+y^2)\quad\cdot y$$

$$\left.\begin{aligned} x\dot{x} &= -xy(x^2+y^2) \\ y\dot{y} &= xy\cdot(x^2+y^2) \end{aligned}\right\} +$$

$$x\dot{x} + y\dot{y} = 0$$

$$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = 0$$

$$\frac{d}{dt}(x^2+y^2) = 0$$

$$x^2+y^2 = a = \text{const.}$$

$$\dot{x} = -a\cdot y \quad \Big| \frac{d}{dt}$$
$$\dot{y} = a\cdot x$$

$$\ddot{x} = -a\dot{y}$$
$$\dot{y} = a\cdot x$$

$$\ddot{x} = -a^2 x$$

$$\ddot{x} + a^2 x = 0$$

$$x = e^{\lambda t}$$

$$\lambda^2 + a^2 = 0$$
$$\lambda^2 = -a^2$$
$$\lambda_{1,2} = \pm\sqrt{-a^2}$$

$$\lambda_1 = ia$$

$$\lambda_2 = -ia$$

$$x(t) = A\cdot\cos(a\cdot t) + B\sin(at)$$

$$y(t) = -\frac{1}{a}\cdot\dot{x}(t) = -\cancel{\frac{1}{a}}\cdot\left(A\cdot -\sin(at)\cdot\cancel{a} + B\cos(at)\cdot\cancel{a}\right)$$

$$y(t) = A\sin(at) - B\cos(at)$$

$$\boxed{\begin{aligned} x(t) &= A\cdot\cos(at) + B\sin(at) \\ y(t) &= A\sin(at) - B\cos(at) \end{aligned}}$$

3) $\vec{r} = \begin{pmatrix} \cos(t) + t\sin(t) \\ \sin(t) - t\cdot\cos(t) \end{pmatrix}$

$$\dot{x} = -\cancel{\sin(t)} + \cancel{\sin(t)} + t\cos(t) = t\cdot\cos(t)$$

$$\dot{y} = \cancel{\cos(t)} - \cancel{\cos(t)} + t\sin(t) = t\sin(t)$$

$$S = \int_0^{\pi} \sqrt{t^2\cos^2(t) + t^2\sin^2(t)}\, dt = \int_0^{\pi} t\cdot dt = \frac{t^2}{2}\Big|_0^{\pi} = \underline{\underline{\frac{\pi^2}{2}}}$$

4.) $\vec{r} = \begin{pmatrix} \cos^3(t) \\ \sin^3(t) \end{pmatrix}$

$$\dot{x} = 3\cos^2(t)\cdot(-\sin(t))$$

$$\dot{y} = 3\sin^2(t)\cdot\cos(t)$$

$$S = \int_0^{\pi/2} \sqrt{\left(9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)\right)}\, dt =$$

$$S = \int_0^{\pi/2} \sqrt{\left(9\sin^2(t)\cdot\cos^2(t)\cdot(\underbrace{\cos^2(t) + \sin^2(t)}_{1})\right)}\, dt$$

$$S = \int_0^{\pi/2} 3\sin(t)\cos(t) = \frac{3}{2}\int_0^{\pi/2} \sin(2t)\, dt$$

$$S = \frac{3}{2} \cdot -\frac{1}{2}\cos(2t)\Big|_0^{\pi/2} = -\frac{3}{4}\cdot(-1-1) = \underline{\underline{\frac{3}{2}}}$$