Übungen Mathematik II am 21. 06. 2024

1.)

Lösen sie die nachstehende DGL 2. Ordnung.

$$y'' + \frac{2}{y} y'^2 + \frac{1}{3} y = 0$$

2.)

Lösen sie nachstehende DGL erster Ordnung.

$$y' = 1 + \cos(y)$$

3.)

Lösen sie nachstehende DGL erster Ordnung.

$$(x^2 - 4)\, y' + (x+2)\, y^2 - 4y = 0$$

4.)

Zeigen sie, dass das folgende Vektorfeld ein Gradientenfeld ist und berechenen sie das sene Linienintegrale entlang des Weges von P = (1,0,0) zu Q = (1,1,0).

$$\vec{F} = \begin{pmatrix} \sin(y) \\ x.\cos(y) + \sin(z) \\ y.\cos(z) \end{pmatrix}$$

1.) $y'' + \frac{2}{y} \cdot y'^2 + \frac{1}{3} \cdot y = 0$

$y' = u$

$y'' = u \cdot \frac{du}{dy}$

$u \cdot \frac{du}{dy} + \frac{2}{y} \cdot u^2 + \frac{1}{3} y = 0 \quad | : u$

$\frac{du}{dy} + \frac{2}{y} \cdot u = -\frac{1}{3} \cdot y \cdot \frac{1}{u}$ "Bernoulli DGL"

$z = u^{1-\alpha} = u^2$

$z' = 2u \cdot u'$

$u' = \frac{1}{2u} \cdot z'$

$z' + \frac{4}{y} \cdot z = -\frac{2}{3} \cdot y$ "Lineare DGL"

a) Homogene

$z' + \frac{4}{y} \cdot z = 0$

$z' = -\frac{4}{y} \cdot z$

$\frac{dz}{z} = -4 \frac{dy}{y}$

$\ln|z| = -4 \ln|y| + \ln|C|$

$z_h = \frac{C}{y^4}$

b) partikulär

$$z_p = C(y) \cdot y^{-4}$$

$$z_p' = C' \cdot y^{-4} + C \cdot (-4) \cdot y^{-5}$$

$$\frac{C'}{y^4} - \cancel{\frac{4C}{y^5}} + \cancel{\frac{4}{y} \cdot \frac{C}{y^4}} = -\frac{2}{3} y$$

$$C' = -\frac{2}{3} y^5$$

$$C = -\frac{2}{3} \cdot \frac{1}{6} y^6 = -\frac{1}{9} \cdot y^6$$

$$z_p = -\frac{1}{9} \cdot y^2$$

$$z = \frac{C}{y^4} - \frac{1}{9} y^2 = u^2 = y'^2$$

$$y' = \sqrt{\frac{C}{y^4} - \frac{1}{9} \cdot y^2}$$

$$\frac{dy}{\sqrt{\frac{C}{y^4} - \frac{1}{9} y^2}} = dx$$

$$\frac{dy}{\frac{\sqrt{C}}{y^2} \cdot \sqrt{1 - \frac{1}{9} \frac{1}{C} y^6}} = dx$$

$$\sin(t) = \frac{1}{3} \cdot \frac{1}{\sqrt{C}} \cdot y^3$$

$$\cos(t)\cdot dt = \cancel{\frac{1}{3}}\cdot\frac{1}{\sqrt{c}}\cdot\cancel{3}\cdot y^2\,dy$$

$$dy = \frac{\sqrt{c}}{y^2}\cdot\cos(t)\,dt$$

$$\frac{\cancel{\sqrt{c}}\cdot\cancel{\cos(t)}\cdot dt}{\cancel{y^2}\cdot\frac{\cancel{\sqrt{c}}}{\cancel{y^2}}\cdot\cancel{\sqrt{1-\sin^2(t)}}} = dx$$

$$dt = dx$$

$$t = (x+D)$$

$$\sin(t) = \sin(x+D)$$

$$\frac{1}{3}\cdot\frac{1}{\sqrt{c}}\cdot y^3 = \sin(x+D)$$

$$y^3 = 3\sqrt{c}\cdot(\sin(x)\cdot\cos(D) + \cos(x)\sin(D))$$

$$\underline{\underline{y = \sqrt[3]{A\cos(x) + B\sin(x)}}}$$

Probe:

$$y'' + \frac{2}{y} y'^2 + \frac{1}{3} y = 0$$

$$y = \left(A\cos(x) + B\sin(x)\right)^{1/3}$$

$$y' = \frac{1}{3} \left(A\cos(x) + B\sin(x)\right)^{-2/3} \cdot \left(-A\sin(x) + B\cos(x)\right)$$

$$y' = \frac{1}{3} \frac{-A\sin(x) + B\cos(x)}{\left(A\cos(x) + B\sin(x)\right)^{2/3}}$$

$$y'' = \frac{1}{3} \frac{\left(-A\cos(x) - B\sin(x)\right) \cdot \left(A\cos(x) + B\sin(x)\right)^{2/3} - \left(-A\sin(x) + B\cos(x)\right)^2 \cdot \frac{2}{3} \cdot (\,)^{-1/3}}{\left(A\cos(x) + B\sin(x)\right)^{4/3}}$$

$$y'' = \frac{1}{3} \cdot \frac{-A\cos(x) - B\sin(x)}{\left(A\cos(x) + B\sin(x)\right)^{2/3}} - \frac{2}{9} \frac{\left(-A\sin(x) + B\cos(x)\right)^2}{\left(A\cos(x) + B\sin(x)\right)^{5/3}}$$

$$\cancel{\frac{1}{3} \cdot \frac{\left(-A\cos(x) - B\sin(x)\right)}{\left(A\cos(x) + B\sin(x)\right)^{2/3}}} - \frac{2}{9} \cdot \cancel{\frac{\left(-A\sin(x) + B\cos(x)\right)^2}{\left(A\cos(x) + B\sin(x)\right)^{5/3}}}$$

$$+ \frac{2}{9} \cdot \cancel{\frac{\left(-A\sin(x) + B\cos(x)\right)^2}{\left(A\cos(x) + B\sin(x)\right)^{5/3}}} + \frac{1}{3} \cdot \cancel{\left(A\cos(x) + B\sin(x)\right)^{1/3}} = 0$$

$$0 = 0$$

2.)

$$y' = \cos(y) + 1$$

"Separierbare DGL"

①

$$\frac{dy}{dx} = \cos(y) + 1$$

$$\frac{dy}{1+\cos(y)} = dx$$

$$\int \frac{dy}{1+\cos(y)} = x + C$$

$$\int \frac{dy}{1+\cos(y)}$$

$$t = \tan\left(\frac{y}{2}\right)$$

$$y = 2\arctan(t)$$

$$dy = \frac{2}{1+t^2} \cdot dt$$

$$\int \frac{2\,dt}{1+t^2 \cdot \left(1 + \frac{1-t^2}{1+t^2}\right)} = 2\int \frac{dt}{1+t^2+1-t^2} = \int dt = t$$

$$\int \frac{dy}{1+\cos(y)} = \tan\left(\frac{y}{2}\right)$$

$$\tan\left(\frac{y}{2}\right)' = \frac{1}{\cos^2\left(\frac{y}{2}\right)} \cdot \frac{1}{2} = \frac{1}{\cos^2\left(\frac{y}{2}\right) + \sin^2\left(\frac{y}{2}\right) + \sin^2\left(\frac{y}{2}\right) - \sin^2\left(\frac{y}{2}\right)} = \frac{1}{1+\cos(y)}$$

$$\tan\left(\frac{y}{2}\right) = x + C$$

$$\frac{y}{2} = \arctan(x+c)$$

$$y = 2\arctan(x+c)$$

Probe:

$$y = 2\arctan(x+c)$$

$$y' = 2\,\frac{1}{1+(x+c)^2}$$

$$\frac{2}{1+(x+c)^2} = \cos(2\cdot\arctan(x+c)) + 1$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\cos^2(\arctan(x+c)) + \underbrace{1 - \sin^2)}\, = 2\cos^2(\arctan(x+c))$$

$$= 2\,\frac{1}{1+(x+c)^2}$$

$$\underline{\underline{0 = 0}}\ \checkmark$$

3.) $(x^2-4)\cdot y' + (x+2)y^2 - 4y = 0$

$$y' - \frac{4}{x^2-4}\cdot y = -\frac{y^2}{(x-2)}$$

$$u = y^{1-\alpha} = \frac{1}{y}$$

$$\underline{\underline{u' = -\frac{y'}{y^2}}} \qquad y' = -y^2 u'$$

$$-y^2\cdot u' - \frac{4}{x^2-4}\cdot y = -\frac{y^2}{(x-2)} \quad \Big|\cdot(-1)$$

$$y^2\cdot u' + \frac{4}{x^2-4}\cdot y = \frac{y^2}{(x-2)}$$

$$u' + \frac{4}{x^2-4}\cdot u = \frac{1}{(x-2)}.$$

1.) homogene DGL:

$$u' + \frac{4}{x^2-4}\cdot u = 0$$

$$u' = \frac{-4}{x^2-4}\cdot u$$

$$\frac{du}{u} = -\frac{4}{x^2-4}\,dx$$

$$\ln(u) = -4\cdot\int\frac{dx}{x^2-4}$$

$$\frac{1}{x^2-4} \rightarrow \frac{A}{x-2} + \frac{B}{x+2} \qquad -4\cdot\int\frac{dx}{x^2-4} = -\int\frac{1}{x-2}\,dx + \int\frac{dx}{x+2}$$

$1 = A\cdot(x+2) + B\cdot(x-2)$ $\qquad = -\ln(x-2) + \ln(x+2) + \ln(C).$

$x=2$: $\quad 1 = 4A \quad A = 1/4$

$x=-2$: $\quad 1 = -4B \quad B = -1/4$

$$\ln(u) = \ln C + \ln(x+2) + \ln(x-2)$$

$$\underline{\underline{u_h = C\cdot\frac{x+2}{x-2}}}$$

2) partikuläre DGL

$$u_p = C(x)\,\frac{x+2}{x-2}$$

$$u_p' = C' \cdot \frac{x+2}{x-2} + C\,\frac{x-2-x-2}{(x-2)^2} = \underline{\underline{C' \cdot \frac{x+2}{x-2} + C\,\frac{-4}{(x-2)^2}}}$$

$$C'\,\frac{x+2}{\cancel{x-2}} - \cancel{\frac{4C}{(x-2)^2}} + \cancel{\frac{4C}{(x-2)^2}} = \frac{1}{\cancel{(x-2)}}$$

$$C' = \frac{1}{x+2}$$

$$\underline{\underline{C = \ln(x+2)}}$$

$$u_p = \ln(x+2) \cdot \frac{x+2}{(x-2)}$$

$$u = C \cdot \frac{x+2}{x-2} + \ln(x+2) \cdot \frac{x+2}{(x-2)}$$

$$\underline{\underline{\frac{1}{y} = \frac{x+2}{x-2} \cdot \left(C + \ln(x+2) \right)}} \qquad C = \frac{1}{y}\,\frac{x-2}{x+2} - \ln(x+2)$$

Probe:

$$-\frac{y'}{y^2} = -\frac{4C}{(x-2)^2} + \frac{1}{x-2} + \ln(x+2) \cdot \left(-\frac{4}{(x-2)^2} \right)$$

$$-\frac{y'}{y^2} = \cancel{+\frac{4\ln(x+2)}{(x-2)^2}} + \frac{1}{x-2} + \cancel{\ln(x+2)\left(\frac{-4}{(x-2)^2}\right)} - \frac{4}{y} \cdot \frac{1}{(x^2-4)} \quad \Big| \cdot y^2$$

$$-y' = \frac{y^2}{(x-2)} - \frac{4y}{(x^2-4)}$$

$$\underline{\underline{y' \cdot (x^2-4) + (x+2) \cdot y^2 - 4y = 0}}$$

4.) $\vec{F} = \begin{pmatrix} \sin(y) \\ x\cos(y) + \sin(z) \\ y\cos(z) \end{pmatrix}$

$$\frac{\partial \phi}{\partial x} = \sin(y)$$

$$\phi = \int \sin(y)\, dx = x\sin(y) + k(y, z)$$

$$\frac{\partial \phi}{\partial y} = x\cancel{\cos(y)} + \frac{\partial k}{\partial y} = \cancel{x\cos(y)} + \sin(z)$$

$$k = \int \sin(z) \cdot dy = \sin(z) \cdot y + h(z)$$

$$\phi = x\sin(y) + y\sin(z) + h(z)$$

$$\frac{\partial \phi}{\partial z} = y\cancel{\cos(z)} + \frac{dh}{dz} = \cancel{y\cos(z)}$$

$$\frac{dh}{dz} = 0 \qquad \underline{\underline{h = \text{Const}}}$$

$$\underline{\underline{\phi = x\sin(y) + y \cdot \sin(z) + C}}$$

$P = (1, 0, 0)$

$Q = (1, 1, 0)$

Wegintegral von P bis Q.

$$\phi(Q) - \phi(P) = 1 \cdot \sin(1) = \underline{\underline{\sin(1)}}$$