Übungen Mathematik II am 01. 06. 2024

1.)

Berechnen sie die folgenden Doppelintegrale.

a.) $\iint_{x=1,\ y=-x}^{x=2,\ y=x} (e^{y} \cosh(x) \quad dy\, dx$ b.) $\iint_{y=0,\ x=0}^{y=\frac{\pi}{2},\ x=\cos(y)} x^2 \sin(y)\, dx\, dy$

2.)

Berechnen sie den Schwerpunkt eines Kreiskegels mit dem Basiskreisradius R und der Hühe H in Zylinderkoordinaten.

3.)

Berechnen sie das Trägheitsmoment eines Kegels, der um die Symmetrieache rotiert in Zylinderkoordinaten.

4.)

Berechnen sie das Trägheitsmoment einer Kugel in Kugelkoordinaten.

1a.)

$$\int_{x=1}^{2} \int_{y=-x}^{x} e^{y} \cosh(x)\, dy\, dx$$

$$\int_{x=1}^{2} e^{y} \Big|_{-x}^{+x} \cdot \cosh(x)\, dx$$

$$\int_{x=1}^{2} \left(e^{x} - e^{-x}\right) \cdot \cosh(x)\, dx = \int_{x=1}^{2} \left(e^{x} - e^{-x}\right) \cdot \left(\frac{1}{2} \cdot \left(e^{x} + e^{-x}\right)\right) dx$$

$$= \frac{1}{2} \int_{x=1}^{2} e^{2x} - e^{-2x}\, dx = \frac{1}{2} \cdot \left(\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right) \Big|_{1}^{2}$$

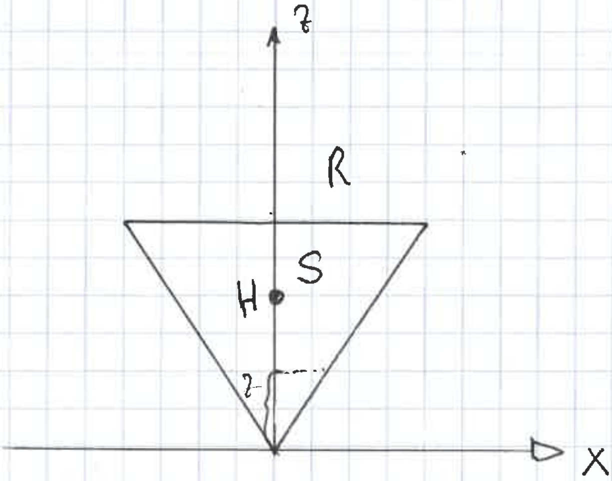
$$= \frac{1}{4} e^{2x} + \frac{1}{4} e^{-2x} = \frac{1}{4} \cdot \left(e^{4} - e^{2} + e^{-4} - e^{-2}\right) \sim \underline{\underline{11,773}}$$

1.b) $\int_{y=0}^{\pi/2} \int_{x=0}^{\cos(y)} x^2 \sin(y) \, dx \, dy$

$$\int_{y=0}^{\pi/2} \frac{x^3}{3} \Big|_0^{\cos(y)} \cdot \sin(y) \, dy = \frac{1}{3} \int_{y=0}^{\pi/2} \sin(y) \cos^3(y) \, dy$$

$$= -\frac{1}{12} \cos^4(y) \Big|_0^{\pi/2} = \underline{\underline{+\frac{1}{12}}}$$

2.) b)



$$M = \rho \int_{r=0}^{\frac{R}{H}\cdot z} \int_{\varphi=0}^{2\pi} \int_{z=0}^{H} r \, dr \, d\varphi \, dz$$

$$M = \rho \cdot 2\pi \int_{z=0}^{H} \frac{r^2}{2} \Big|_0^{\frac{R}{H}\cdot z} dz$$

$$M = \cancel{2}\pi\rho \cdot \frac{1}{\cancel{2}} \int_{z=0}^{H} \frac{R^2}{H^2} z^2 \, dz = \rho\pi \frac{R^2}{\cancel{H^2}} \frac{H^{\cancel{3}}}{3}$$

$$M = \frac{\rho \cdot R^2\pi \cdot H}{3}$$

$$z_S = \frac{1}{M} \int_{r=0}^{\frac{R}{H}\cdot z} \int_{\varphi=0}^{2\pi} \int_{z=0}^{H} \rho \cdot z \cdot r \, dr \, d\varphi \, dz$$

$$z_S = \frac{3}{\cancel{\rho} R^2 \cancel{\pi} H} \cdot \cancel{\rho} \cdot \cancel{2}\cancel{\pi} \frac{1}{\cancel{2}} \int_{z=0}^{H} \frac{R^2}{H^2} \cdot z^3 \, dz = \frac{3}{\cancel{R^2}\cdot \cancel{H}} \cdot \frac{\cancel{R^2}}{\cancel{H^2}} \cdot \frac{H^{\cancel{4}}}{4} = \frac{3H}{4}$$

3.) $I = \int\int\int \rho \cdot r^2 \cdot \rho \, dV$

$$I = \rho \int_{r=0}^{\frac{R}{H}\cdot z} \int_{\varphi=0}^{2\pi} \int_{z=0}^{H} r^2 \cdot r \, dr \, d\varphi \, dz$$

$$I = 2\pi \cdot \rho \int_{z=0}^{H} \frac{r^4}{4} \Big|_0^{\frac{R}{H}\cdot z} dz$$

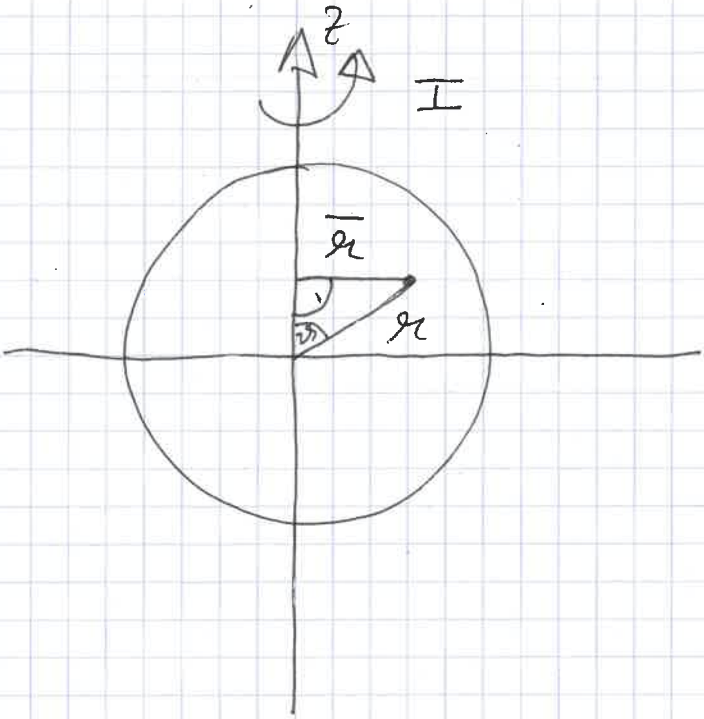
$$I = 2\pi \cdot \frac{1}{4}\rho \int_{z=0}^{H} \frac{R^4}{H^4} \cdot z^4 \, dz =$$

$$I = \rho \frac{\pi}{2} \cdot \frac{R^4}{H^4} \frac{z^5}{5} \Big|_0^H = \frac{1}{10} \pi \frac{R^4}{\cancel{H^4}} H^{\cancel{5}}$$

$$I = \frac{1}{10} \Big(\underbrace{\frac{\rho \pi R^2 \cdot H}{3}}_{M} \Big) \cdot 3 R^2$$

$$\underline{\underline{I_{Kegel} = \frac{3}{10} M R^2}}$$

4.)



$$I = \iiint \rho \cdot \bar{r}^2 \, dV$$

$$dV = r^2 \cdot \sin\vartheta \, dr \, d\varphi \, d\vartheta$$

$$\bar{r} = r \cdot \sin\vartheta$$

$$I = \rho \int\limits_{r=0}^{R} \cdot \int\limits_{\varphi=0}^{2\pi} \int\limits_{\vartheta=0}^{\pi} r^2 \sin^2\vartheta \; r^2 \sin\vartheta \, dr \, d\varphi \, d\vartheta$$

$$I = \rho \cdot 2\pi \; \frac{R^5}{5} \int\limits_{\vartheta=0}^{\pi} \sin^3\vartheta \, d\vartheta$$

$$I = \frac{2}{5} \rho\pi R^5 \int\limits_{\vartheta=0}^{\pi} \sin\vartheta \left(1 - \cos^2\vartheta\right) d\vartheta$$

$$I = \frac{2}{5} \rho\pi R^5 \cdot \left(-\cos\vartheta + \frac{\cos^3\vartheta}{3}\right)\Big|_0^{\pi} = \frac{2}{5} \rho\pi R^5 \cdot \left(2 - \frac{2}{3}\right)$$

$$I = \frac{2}{5} \rho\pi R^5 \cdot \frac{4}{3} = \underbrace{\rho \cdot \underbrace{\frac{4\pi R^3}{3}}_{V}}_{M} \cdot \frac{2}{5} R^2 = \underline{\underline{\frac{2}{5} M R^2}}$$