Übungen Mathematik II am 27. 04. 2024

1.)

Lösen sie das nachstehende Differentialgleichungssystem für die gegebenen Anfangs-bedingungen. Wie sieht die Bahn der Bewegung aus, wenn die Zeit t gegen Unendlich geht.

$$\dot{x} + x - m.y = \sin(m.t)$$
$$\dot{y} + y + m.x = \cos(m.t)$$

$$x(0) = 1$$
$$y(0) = 0$$

2.)

Lösen sie das folgende Differentialgleichungssystem für die gegebenen Anfangsbedingungen.

$$\dot{x} = 5x + 4y - 5t^2 + 6t + 25$$
$$\dot{y} = x + 2y - t^2 + 2t + 4$$

$$x(0) = 0$$
$$y(0) = 0$$

3.)

Lösen sie das folgende Differentialgleichungssystem für die gegebenen Anfangsbedingungen.

$$\dot{x} = y$$
$$\dot{y} = z$$
$$\dot{z} = 4x - 4y + z$$

4.)

Lösen sie das folgende Differentialgleichungssystem für die gegebenen Anfangsbedingungen.

$$\dot{x} = x$$
$$\dot{y} = 2x + y - 2z$$
$$\dot{z} = 3x + 2y + z$$

1.)

$$\dot{x} + x - my = \sin(mt) \qquad y = \frac{1}{m}\cdot(\dot{x} + x - \sin(mt))$$

$$\dot{y} + y + mx = \cos(mt) \qquad \dot{y} = \cos(mt) - mx - y$$

1) $\frac{d}{dt}$:

$$\ddot{x} + \dot{x} - m\dot{y} = m\cdot\cos(mt)$$

$$\ddot{x} + \dot{x} - m\cdot(\cos(mt) - mx - y) = m\cdot\cos(mt)$$

$$\ddot{x} + \dot{x} - m\cdot(\cos(mt) - mx - \frac{1}{m}\cdot(\dot{x} + x - \sin(mt))) = m\cos(mt)$$

$$\ddot{x} + \dot{x} - m\cdot\cos(mt) + m^2 x + \dot{x} + x - \sin(mt) = m\cdot\cos(mt)$$

$$\underline{\underline{\ddot{x} + 2\dot{x} + (m^2+1)\cdot x = 2m\cos(mt) + \sin(mt)}}$$

1.) homogene DGL:

$$\ddot{x} + 2\dot{x} + (m^2+1)\cdot x = 0$$

$$x = e^{rt}$$

$$r^2 + 2r + m^2 + 1 = 0$$

$$r_{1,2} = -1 \pm \sqrt{1 - m^2 - 1}$$

$$r_{1,2} = -1 \pm \sqrt{-m^2}$$

$$\underline{\underline{r_1 = -1 + im}}$$

$$\underline{\underline{r_2 = -1 - im}}$$

$$\underline{\underline{x_h(t) = A\cdot e^{-t}\cdot\cos(mt) + B\cdot e^{-t}\sin(mt)}}$$

$$X_p(t) = a \cdot \cos(\omega t) + b \sin(\omega t).$$

$$\dot{X}_p(t) = -a\,\omega \sin(\omega t) + b\,\omega \cdot \cos(\omega t)$$

$$\ddot{X}_p(t) = -a\omega^2 \cos(\omega t) - b\omega^2 \sin(\omega t).$$

$$-a\omega^2 \cos(\omega t) - b\omega^2 \sin(\omega t) - 2a\omega \sin(\omega t) + 2b\omega \cos(\omega t) + a\omega^2 \cos(\omega t) + b\omega^2 \sin(\omega t) + a\cos(\omega t) + b\sin(\omega t)$$
$$= 2\omega \cdot \cos(\omega t) + \sin(\omega t).$$

$$\cos(\omega t) \cdot \left[-\cancel{a\omega^2} + 2b\omega + \cancel{a\omega^2} + a\right] = 2\omega \cdot \cos(\omega t)$$
$$\sin(\omega t) \cdot \left[-\cancel{b\omega^2} - 2a\omega + \cancel{b\omega^2} + b\right] = 1 \cdot \sin(\omega t)$$

$$b \cdot 2\omega + a = 2\omega \quad | \cdot 2\omega$$
$$b - a \cdot 2\omega = 1$$

$$\left.\begin{aligned} 4\omega^2 \cdot b + 2\omega a &= 4\omega^2 \\ b - 2\omega a &= 1 \end{aligned}\right\} +$$

$$b \cdot (4\omega^2 + 1) = 4\omega^2 + 1$$

$$\underline{\underline{b = 1}} \qquad \underline{\underline{a = 0}}$$

$$\underline{\underline{X_p(t) = \sin(\omega t)}}$$

$$\underline{\underline{X(t) = A \cdot e^{-t} \cdot \cos(\omega t) + B e^{-t} \sin(\omega t) + \sin(\omega t)}}$$

$$y(t) = \frac{1}{\omega} \cdot (\dot{x} + x - \sin \omega t).$$

$$\dot{x}(t) = -Ae^{-t} \cos(\omega t) - Ae^{-t} \cdot \omega \cdot \sin(\omega t) - Be^{-t} \sin(\omega t) + Be^{-t} \cdot \omega \cos(\omega t) + \omega \cdot \cos(\omega t).$$

$$y(t) = \frac{1}{\omega} \cdot \left[-\cancel{A \cdot e^{-t} \cos(\omega t)} - Ae^{-t}\omega \cdot \sin(\omega t) - \cancel{Be^{-t} \sin(\omega t)} + Be^{-t}\omega \cos(\omega t) + \omega \cos(\omega t)\right.$$
$$\left. + \cancel{Ae^{-t} \cos(\omega t)} + \cancel{Be^{-t} \sin(\omega t)} + \cancel{\sin(\omega t)} - \cancel{\sin(\omega t)}\right]$$

$$\underline{\underline{y(t) = -Ae^{-t} \sin(\omega t) + Be^{-t} \cos(\omega t) + \cos(\omega t)}}$$

(3)

$$\boxed{\begin{aligned} x(t) &= A\cdot e^{-t}\cos(\omega t) + B e^{-t}\sin(\omega t) + \sin(\omega t) \\ y(t) &= -A e^{-t}\sin(\omega t) + B e^{-t}\cos(\omega t) + \cos(\omega t) \end{aligned}}$$

$t \to \infty$ Kreis $e^{-t} \to 0$

$$x(t) = \sin(\omega t)$$

$$y(t) = \cos(\omega t)$$

$$x^2 + y^2 = \sin^2(\omega t) + \cos^2(\omega t) = 1$$

$$x^2 + y^2 = 1$$ „Einheitskreis"

Anfangsbedingungen: $x(0) = 1$

$y(0) = 0.$

$1 = A$

$0 = B + 1 \quad \to \quad B = -1$

$$\boxed{\begin{aligned} x(t) &= e^{-t}\cdot\cos(\omega t) - e^{-t}\sin(\omega t) + \sin(\omega t) \\ y(t) &= -e^{-t}\cdot\sin(\omega t) - e^{-t}\cos(\omega t) + \cos(\omega t) \end{aligned}}$$

2.) $\dot{x} = 5x + 4y - 5t^2 + 6t + 25$ (I) $x(0) = 0$

$\dot{y} = x + 2y - t^2 + 2t + 4$ (II) $y(0) = 0$

aus (II): $x = \dot{y} - 2y + t^2 - 2t - 4$

$\dot{x} = \ddot{y} - 2\dot{y} + 2t - 2$

Einsetzen in (I).

$$\ddot{y} - 2\dot{y} + 2t - 2 = 5\dot{y} - 10y \cancel{+5t^2} - 10t - 20 + 4y \cancel{-5t^2} + 6t + 25$$

$$\ddot{y} - 7\dot{y} + 6y = -6t + 7$$

$+7$

a.) Homogene DGL

$$\ddot{y} - 7\dot{y} + 6y = 0$$

$$r^2 - 7r + 6 = 0$$

$$r_{1,2} = \frac{7}{2} \pm \sqrt{\frac{49}{4} - \frac{24}{4}}$$

$$r_1 = \frac{7}{2} \pm \frac{5}{2}$$

$$r_1 = 6$$

$$r_2 = 1$$

$$y_h = A \cdot e^{6t} + Be^{t}$$

$y_p = K_0 + K_1 t$

$\dot{y}_p = K_1$

$\ddot{y}_p = 0$

$$-7K_1 + 6K_0 + 6K_1 t = -6t + 7$$

$K_1 = -1$ $K_0 = 0$ $\Rightarrow$ $y_p = -t$

$$\underline{\underline{y(t) = A e^{6t} + Be^{t} - t}}$$

$$x = \overset{\circ}{y} - 2y + t^2 - 2t - 4$$

$$x = 6Ae^{6t} + Be^{t} - 1 - 2Ae^{6t} - 2Be^{t} \cancel{+2t} + t^2 \cancel{-2t} - 4$$

$$\underline{\underline{x(t) = 4Ae^{6t} - Be^{t} + t^2 - 5}}$$

$x(0) = 0$

$y(0) = 0$

$$\left.\begin{array}{l} 4A - B = 5 \\ \underline{A + B = 0} \end{array}\right\} \quad \begin{array}{l} A = 1 \\ B = -1 \end{array}$$

$$\boxed{\begin{array}{l} x(t) = 4e^{6t} + e^{t} + t^2 - 5 \\ y(t) = e^{6t} - e^{t} - t \end{array}}$$

3.) $\dot{x} = y$ (I)

$\dot{y} = z$ (II)

$\dot{z} = 4x - 4y + z$ (III)

$\frac{d}{dt}(I) \Rightarrow \ddot{x} = \dot{y} = z$

$\frac{d}{dt}(-''-): \dddot{x} = \dot{z}$

$\dddot{x} = 4x - 4y + z$

$\dddot{x} = 4x - 4y + \dot{y}$

$\dddot{x} = 4x - 4\dot{x} + \ddot{x}$

$\dddot{x} - \ddot{x} + 4\dot{x} - 4x = 0$

$x(t) = e^{\lambda t}$

$\lambda^3 - \lambda^2 + 4\lambda - 4 = 0$

$\underline{\underline{\lambda_1 = 1}}$

$(\lambda^3 - \lambda^2 + 4\lambda - 4) : (\lambda - 1) = \underline{\underline{\lambda^2 + 4}}$

$\underline{-\lambda^3 + \lambda^2}$

$4\lambda - 4$

$\underline{-4\lambda + 4}$

0 Rest

$\underline{\underline{\lambda_2 = 2i}}$

$\underline{\underline{\lambda_3 = -2i}}$

$$x(t) = C_1 e^t + C_2 \cos(2t) + C_3 \sin(2t)$$

$$y(t) = C_1 \cdot e^t - 2C_2 \sin(2t) + 2C_3 \cos(2t)$$

$$z(t) = C_1 e^t - 4C_2 \cos(2t) - 4C_3 \sin(2t)$$

4) I. $\dot{x} = x$

II. $\dot{y} = 2x + y - 2z$

III. $\dot{z} = 3x + 2y + z$

I.)

$$\dot{x} = x$$

$$\frac{dx}{dt} = x$$

$$\frac{dx}{x} = dt$$

$$\ln|x| = t + \ln(c)$$

$$\underline{\underline{x(t) = C \cdot e^{t}}}$$

II. $\dot{y} = 2x + y - 2z = y - 2z + 2C \cdot e^{t}$

III. $\dot{z} = 3x + 2y + z = 2y + z + 3C \cdot e^{t}$

Aus II.)

$$\dot{y} = y - 2z + 2C e^{t}$$

$$z = C e^{t} + \frac{y}{2} - \frac{\dot{y}}{2}$$

$$\dot{z} = C e^{t} + \frac{\dot{y}}{2} - \frac{\ddot{y}}{2}$$

Einsetzen in III.)

$$\cancel{C e^{t}} + \frac{\dot{y}}{2} - \frac{\ddot{y}}{2} = 2y + \cancel{C e^{t}} + \frac{y}{2} - \frac{\dot{y}}{2} + 3C e^{t}$$

$$-\frac{\ddot{y}}{2} + \dot{y} - \frac{5}{2} y = 3C e^{t}$$

$$\underline{\underline{\ddot{y} - 2\dot{y} + 5y = -6C e^{t}}}$$

a) homogène.

$$\ddot{y} - 2\dot{y} + 5y = 0$$

$$r^2 - 2r + 5 = 0$$

$$r_{1,2} = 1 \pm \sqrt{1-5}$$

$$r_1 = 1 + 2i$$

$$r_2 = 1 - 2i$$

$$y_h(t) = e^t \cdot (A\cos(2t) + B\sin(2t))$$

b) particulière

$$y_p = D \cdot e^t$$

$$\dot{y}_p = De^t$$

$$\ddot{y}_p = De^t$$

$$D - 2D + 5D = -6C$$

$$4D = -6C$$

$$D = -\frac{3}{2} \cdot C$$

$$y_p = -\frac{3}{2} \cdot C e^t$$

$$y(t) = e^t \cdot (A\cos(2t) + B\sin(2t)) - \frac{3}{2} \cdot C \cdot e^t$$

$$z(t) = Ce^t + \frac{y}{2} - \frac{\dot{y}}{2}$$

$$= Ce^t + \frac{A}{2}e^t\cos(2t) + \frac{B}{2}e^t\cdot\sin(2t) - \cancel{\frac{3}{4}\cdot Ce^t}$$

$$- \frac{1}{2}\cdot\left(Ae^t\cos(2t) + Be^t\sin(2t) - 2Ae^t\sin(2t) + 2Be^t\cos(2t)\right) + \cancel{\frac{3}{4}\cdot C\cdot e^t}$$

$$z(t) = Ce^t + \cancel{\frac{A}{2}e^t\cos(2t)} + \cancel{\frac{B}{2}e^t\sin(2t)} - \cancel{\frac{A}{2}e^t\cos(2t)} - \cancel{\frac{B}{2}e^t\sin(2t)}$$

$$+ Ae^t\sin(2t) - Be^t\cos(2t)$$

$$\underline{\underline{z(t) = Ae^t\sin(2t) - Be^t\cos(2t) + Ce^t}}$$

$$\boxed{\begin{aligned} x(t) &= C\cdot e^t \\ y(t) &= Ae^t\cos(2t) + Be^t\sin(2t) - \frac{3}{2}\cdot C\cdot e^t \\ z(t) &= Ae^t\sin(2t) - Be^t\cos(2t) + Ce^t \end{aligned}}$$

Probe:

$$\dot{x} = x$$

$$C\cdot e^t = C\cdot e^t \quad \checkmark$$

$$\dot{y} = 2x + y - 2z$$

$$\cancel{Ae^t\cos(2t)} - \cancel{2Ae^t\sin(2t)} + \cancel{Be^t\sin(2t)} + \cancel{2Be^t\cos(2t)} - \cancel{\frac{3}{2}Ce^t} =$$

$$\cancel{2Ce^t} + \cancel{Ae^t\cos(2t)} + \cancel{Be^t\sin(2t)} - \cancel{\frac{3}{2}Ce^t} - \cancel{2Ae^t\sin(2t)} + \cancel{2Be^t\cos(2t)} - \cancel{2Ce^t}$$

$$0 = 0 \quad \checkmark$$

$$\dot{z} = 3x + 2y + z$$

$$\cancel{Ae^t\sin(2t)} + \cancel{2Ae^t\cos(2t)} - \cancel{Be^t\cos(2t)} + \cancel{2Be^t\sin(2t)} + \cancel{C\cdot e^t} =$$

$$\cancel{3\cdot Ce^t} + \cancel{2Ae^t\cos(2t)} + \cancel{2Be^t\sin(2t)} - \cancel{3Ce^t} + \cancel{Ae^t\sin(2t)} - \cancel{Be^t\cos(2t)} + \cancel{Ce^t}$$

$$0 = 0 \quad \checkmark$$