Übungen Mehrdimensionale Analysis am 06. 04. 2024

1.)

Lösen Sie nachstehende Differentialgleichung.

$$(\cos(x+y^2) + 3y)\,dx + (2y\cos(x+y^2) + 3x)\,dy = 0$$

2.)

Lösen Sie nachstehende Differentialgleichung.

$$-2xy\,dx + (3x^2 - y^2)\,dy = 0$$

3.)

Lösen Sie nachstehende Differentialgleichung.

$$\cos(x)\,dx + (4y\,e^{-y} + \sin(x))\,dy = 0$$

4.)

Lösen Sie nachstehende Differentialgleichung.

$$3x^2.(y-x)^2\,dy + (\sin(x) - x\cos(x) - 3x^2.(y-x)^2)\,dx = 0$$

1.) $$\underbrace{\left[\cos(x+y^2)+3y\right]}_{M(x,y)}\cdot dx+\left(\underbrace{2y\cdot\cos(x+y^2)+3x}_{N(x,y)}\right)dy=0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=-\sin(x+y^2)\cdot 2y+3\\ \frac{\partial N}{\partial x}&=2y\cdot-\sin(x+y^2)+3\end{aligned}\right\}\text{ exakte DGL.}$$

$$\frac{\partial u}{\partial x}=\cos(x+y^2)+3y$$

$$u=\int\left[\cos(x+y^2)+3y\right]dx$$

$$\underline{\underline{u=\sin(x+y^2)+3xy+k(y)}}$$

$$\frac{\partial u}{\partial y}=\cancel{\cos(x+y^2)\cdot 2y}+\cancel{3x}+\frac{dk}{dy}=\cancel{2y\cdot\cos(x+y^2)}\,\cancel{+3x}$$

$$\frac{dk}{dy}=0$$

$$\underline{\underline{k=\text{const.}}}$$

$$u=\sin(x+y^2)+3xy+\text{const}=\text{const.}$$

$$\underline{\sin(x+y^2)+3xy=C}$$

Probe: $\cos(x+y^2)(1+2yy')+3y+3xy'=0$

$$(2y\cdot\cos(x+y^2)+3x)\cdot y'+\cos(x+y^2)+3y=0$$

$$\left[\cos(x+y^2)+3y\right]\cdot dx+\left[2y\cos(x+y^2)+3x\right]dy=0$$

2.) $(3x^2 - y^2) \cdot y' = 2xy$

$$y' = \frac{2xy}{3x^2-y^2}$$

$$\underbrace{-2xy}_{M(x,y)} \, dx + \underbrace{(3x^2 - y^2)}_{N(x,y)} \cdot dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -2x \\ \frac{\partial N}{\partial x} &= 6x \end{aligned}\right\} \text{ nicht existiert.}$$

$\underline{\underline{F = F(x)}}$ $\quad \frac{1}{F} \frac{dF}{dx} = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$

$$\frac{1}{F} \cdot \frac{dF}{dx} = \frac{1}{3x^2-y^2} \left(-2x - 6x \right) = \underline{\underline{-\frac{8x}{3x^2-y^2}}}$$

$\underline{\underline{F(X): \text{ existiert nicht}}}$

$\underline{\underline{F = F(y):}}$

$$\frac{1}{F} \cdot \frac{dF}{dy} = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$\frac{1}{F} \cdot \frac{dF}{dy} = -\frac{1}{2xy} \cdot \left(6x + 2x \right)$$

$$\frac{1}{F} \cdot \frac{dF}{dy} = -\frac{1}{2xy} \cdot 8x = -\frac{4}{y}$$

$$\frac{dF}{F} = -\frac{4}{y} \, dy$$

$$\ln(F) = -4 \ln(y)$$

$$\underline{\underline{F = \frac{1}{y^4}}}$$

$$-\frac{2xy}{y^4}\,dx + \left(\frac{3x^2}{y^4} - \frac{1}{y^2}\right) dy = 0$$

$$\underbrace{-\frac{2x}{y^3}}_{M} \cdot dx + \left(\underbrace{3\frac{x^2}{y^4} - \frac{1}{y^2}}_{N}\right) \cdot dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -2x \cdot \frac{-3}{y^4} = \frac{6x}{y^4} \\ \frac{\partial N}{\partial x} &= \frac{6x}{y^4} \end{aligned}\right\} \quad \underline{\text{exakt DGL}}$$

$$\frac{\partial u}{\partial x} = -\frac{2x}{y^3}$$

$$u = -\frac{x^2}{y^3} + k(y)$$

$$\frac{\partial u}{\partial y} = \cancel{-x^2 \cdot \frac{-3}{y^4}} + \frac{dk}{dy} = \cancel{\frac{3x^2}{y^4}} - \frac{1}{y^2}$$

$$\frac{dk}{dy} = -\frac{1}{y^2}$$

$$\underline{k = \frac{1}{y}}$$

$$u = -\frac{x^2}{y^3} + \frac{1}{y} = C$$

$$\underline{\underline{\frac{x^2}{y^3} - \frac{1}{y} = C}}$$

Als Euler-homogene DGL

$$y' = \frac{2xy}{3x^2 - y^2}$$

$$y' = \frac{x^2 \cdot 2\frac{y}{x}}{x^2 \cdot \left(3 - \frac{y^2}{x^2}\right)} = \frac{2\,{}^{y}/_{x}}{3 - {}^{y^2}/_{x^2}}$$

$$u = \frac{y}{x}$$

$$y = u \cdot x$$

$$y' = u'x + u$$

$$u'x + u = \frac{2u}{3 - u^2}$$

$$u'x = \frac{2u}{3-u^2} - u = \frac{2u - 3u + u^3}{3 - u^2}$$

$$u'x = \frac{u^3 - u}{3 - u^2}$$

$$\frac{3 - u^2}{u^3 - u} \cdot du = \frac{dx}{x}$$

$$\frac{3 - u^2}{u \cdot (u-1) \cdot (u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}$$

$$3 - u^2 = A \cdot (u^2 - 1) + B \cdot u \cdot (u+1) + C \cdot u \cdot (u-1)$$

$u = 0$: $\quad 3 = A \cdot (-1) \qquad \underline{\underline{A = -3}}$

$u = 1$: $\quad 2 = B \cdot 2 \qquad \underline{\underline{B = 1}}$

$u = -1$: $\quad 2 = C \cdot (-1) \cdot (-2) \qquad \underline{\underline{C = 1}}$

$$\left(-\frac{3}{u}+\frac{1}{u+1}+\frac{1}{u-1}\right)\cdot du = \frac{dx}{x}$$

$$-3\ln|u| + \ln|u+1| + \ln|u-1| = \ln|x| + \ln|c|$$

$$\ln\left(\frac{u^2-1}{u^3}\right) = \ln(c\cdot x)$$

$$\frac{u^2-1}{u^3} = C\cdot X$$

$$\frac{\frac{y^2}{x^2}-1}{\frac{y^3}{x^3}} = C\cdot x$$

$$\frac{y^2-x^2}{\cancel{x^2}}\cdot\frac{x^{\cancel{3}}}{y^3} = C\cdot x$$

$$-\frac{x^2}{y^3}+\frac{1}{y} = C$$

$$\frac{x^2}{y^3}-\frac{1}{y} = C_1$$

Probe: $\frac{x^2}{y^3}-\frac{1}{y} = C_1 \;\Big|\; \frac{d}{dx}$

$$\frac{2x}{y^3}+x^2\cdot\frac{-3}{y^4}\cdot y'+\frac{1}{y^2}\cdot y' = 0 \;\Big|\cdot y^4$$

$$2xy + (y^2-3x^2)\cdot y' = 0$$

$$2xy\,dx + (y^2-3x^2)\cdot dy = 0$$

3.) $\underbrace{\cos(x)}_{M(x,y)} \cdot dx + \underbrace{(4y \cdot e^{-y} + \sin(x))}_{N(x,y)} \cdot dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 0 \\ \frac{\partial N}{\partial x} &= \cos(x) \end{aligned}\right\} \text{nicht exakt.}$$

$F = F(x)$

$$\frac{1}{F} \cdot \frac{dF}{dx} = \frac{1}{N} \cdot \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

$$\frac{1}{F} \cdot \frac{dF}{dx} = \frac{1}{4ye^{-y} + \sin(x)} \cdot \left(0 - \cos(x) \right)$$ nicht möglich.

$F(x)$: existiert nicht.

$F = F(y)$:

$$\frac{1}{F} \cdot \frac{dF}{dy} = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$\frac{1}{F} \cdot \frac{dF}{dy} = \frac{1}{\cos(x)} \cdot \left(\cos(x) \right)$$

$$\frac{1}{F} \cdot \frac{dF}{dy} = 1$$

$$\frac{dF}{F} = dy$$

$$\ln(F) = y$$

$$F = e^{y}$$

$$\underbrace{e^{y} \cdot \cos(x)}_{M(x,y)}\, dx + \big(\underbrace{4y + \sin(x) \cdot e^{y}}_{N(x,y)}\big)\, dy = 0$$

$$\frac{\partial u}{\partial x} = e^{y} \cdot \cos(x)$$

$$u = e^{y} \cdot \sin(x) + k(y)$$

$$\frac{\partial u}{\partial y} = \cancel{e^{y} \cdot \sin(x)} + \frac{dk}{dy} = 4y + \cancel{\sin(x) \cdot e^{y}}$$

$$\frac{dk}{dy} = 4y$$

$$\underline{\underline{k = 2y^2}}$$

$$u = e^{y} \cdot \sin(x) + 2y^2 = C$$

$$\underline{\underline{e^{y} \cdot \sin(x) + 2y^2 = C}}$$

4.)

$$y' \cdot 3x^2 \cdot (y-x)^2 + \sin(x) - x\cos(x) - 3x^2 \cdot (y-x)^2 = 0.$$

$$\Big[\underbrace{\sin(x) - x\cos(x) - 3x^2 \cdot (y-x)^2}_{M(x,y)}\Big] \cdot dx + \Big[\underbrace{3x^2 \cdot (y-x)^2}_{N(x,y)}\Big] \cdot dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -3x^2 \cdot 2 \cdot (y-x) = -6x^2 \cdot (y-x) \\ \frac{\partial N}{\partial x} &= 6x \cdot (y-x)^2 + 3x^2 \cdot 2 \cdot (y-x) \cdot (-1) \end{aligned}\right\} \text{kein exakte DGL}$$

Annahme: $F = F(x)$

$$\frac{1}{F} \cdot \frac{dF}{dx} = \frac{1}{N} \cdot \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

$$\frac{1}{F} \cdot \frac{dF}{dx} = \frac{1}{3x^2 \cdot \cancel{(y-x)^2}} \left(\cancel{-6x^2(y-x)} - 6x \cdot \cancel{(y-x)^2} + \cancel{6x^2(y-x)} \right)$$

$$\frac{1}{F} \cdot \frac{dF}{dx} = \frac{-6x}{3x^2}$$

$$\frac{1}{F} \cdot \frac{dF}{dx} = -\frac{2}{x}.$$

$$\frac{dF}{F} = -2 \frac{dx}{x}$$

$$\ln(F) = -2\ln(x)$$

$$F = \frac{1}{x^2}$$

$F = F(y)$: geht nicht

$$\underbrace{\left[\frac{1}{x^2}\sin(x) - \frac{1}{x}\cos(x) - 3\cdot(y-x)^2\right]}_{M(x,y)} dx + \underbrace{3(y-x)^2}_{N(x,y)} dy = 0.$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -6\cdot(y-x) \\ \frac{\partial N}{\partial x} &= 6(y-x)\cdot(-1) \end{aligned}\right\} \quad \underline{\text{exakte DGL}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2}\cdot\sin(x) - \frac{1}{x}\cos(x) - 3(y-x)^2$$

$$u = \int \underset{g'}{\frac{1}{x^2}}\underset{h}{\sin(x)} - \frac{1}{x}\cos(x) - 3\cdot(y-x)^2 \; dx$$

$$u = -\frac{1}{x}\cdot\sin(x) - \cancel{\int -\frac{1}{x}\cdot\cos(x)\,dx} - \cancel{\int \frac{1}{x}\cos(x)\,dx} + 3\cdot(y-x)^3\cdot\frac{1}{3} + k(y)$$

$$\underline{\underline{u = -\frac{1}{x}\sin(x) + (y-x)^3 + k(y)}}$$

$$\frac{\partial u}{\partial y} = \cancel{3\cdot(y-x)^2} + \frac{dk}{dy} = \cancel{3\cdot(y-x)^2}$$

$$k = \text{const}$$

$$-\frac{1}{x}\sin(x) + (y-x)^3 = C$$

$$\underline{\underline{(y-x)^3 - \frac{1}{x}\sin(x) = C}}$$

Probe $\quad 3\cdot(y-x)^2\cdot(y'-1) + \frac{1}{x^2}\sin(x) - \frac{1}{x}\cos(x) = 0$

$$\underline{3x^2\cdot(y-x)^2\cdot y' + \sin(x) - x\cos(x) - 3x^2\cdot(y-x)^2 = 0}$$

Oder 2. Weg:

$$\frac{\partial u}{\partial y} = 3\cdot(y-x)^2$$

$$u = 3\cdot\int (y-x)^2\cdot dy$$

$$u = 3\cdot\frac{(y-x)^3}{3} + h(x)$$

$$\underline{\underline{u = (y-x)^3 + h(x)}}$$

$$\frac{\partial u}{\partial x} = \cancel{3\cdot(y-x)^2\cdot(-1)} + \frac{dh}{dx} = \frac{1}{x^2}\sin(x) - \frac{1}{x}\cos(x)\ \cancel{-3(y-x)^2}$$

$$\frac{dh}{dx} = \frac{1}{x^2}\cdot\sin(x) - \frac{1}{x}\cdot\cos(x)$$

$$h = \int \underset{u'}{\frac{1}{x^2}}\cdot\underset{v}{\sin(x)}\,dx - \int\frac{1}{x}\cos(x)\,dx$$

$$h = -\frac{1}{x}\sin(x)\ \cancel{-\int -\frac{1}{x}\cdot\cos(x)\,dx}\ \cancel{-\int\frac{1}{x}\cos(x)\,dx}$$

$$\underline{\underline{h = -\frac{1}{x}\cdot\sin(x)}}$$

$$\underline{\underline{(y-x)^3 - \frac{1}{x}\cdot\sin(x) = C}}$$

Odu:

$$y' \cdot 3x^2 \cdot (y-x)^2 + \sin(x) - x\cos(x) - 3x^2 \cdot (y-x)^2 = 0$$

Subst.: $u = y - x$

$u' = y' - 1$

$$(u'+1) \cdot (3x^2)(y-x)^2 + \sin(x) - x\cos(x) - 3x^2 \cdot (y-x)^2 = 0$$

$$u' \, 3x^2 \cdot (\overbrace{y-x}^{u^2})^2 + \cancel{3x^2 \cdot (y-x)^2} + \sin(x) - x\cos(x) - \cancel{3x^2 \cdot (y-x)^2} = 0$$

$$u' \cdot 3x^2 \cdot u^2 + \sin(x) - x\cos(x) = 0$$

$$u^2 \cdot u' = \frac{1}{3x^2} \cdot (x\cos(x) - \sin(x))$$

$$u^2 \cdot du = \frac{1}{3} \cdot \left(\frac{1}{x}\cos(x) - \frac{1}{x^2}\sin(x)\right) dx$$

$$\frac{u^3}{3} = \frac{1}{3} \cdot \left(\frac{1}{x} \cdot \sin(x) - \cancel{\int -\frac{1}{x^2} \sin(x)\, dx} - \cancel{\int \frac{1}{x^2}\sin(x)\, dx}\right)$$

$$u^3 = \frac{1}{x}\sin(x) + C$$

$$\underline{\underline{(y-x)^3 - \frac{1}{x}\sin(x) = C}}$$