Übungen Mathematik II am 07. 06. 2024

1.)
Berechnen sie die folgenden Doppelintegrale in Polarkoordinaten.

a.) $f(x,y) = e^{-(x^2+y^2)}$

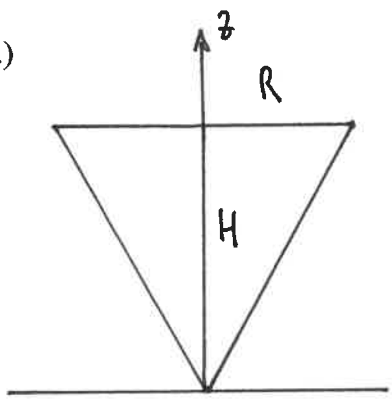
Integrationsbereich zwischen den Kreisen mit dem Radius 1 und dem Radius 2.

b.) $f(x,y) = (x^2 + y^2)^2$

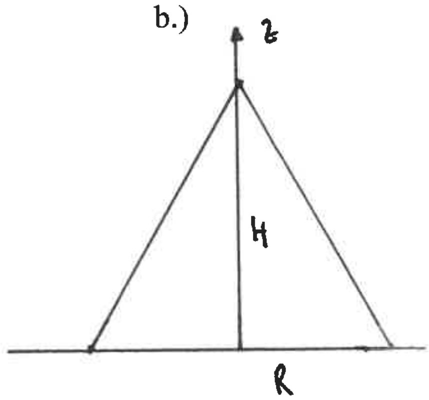
Integrationsbereich Kreis mit dem Radius 2 und $y \geq 0$

2.)
Berechnen sie das Volumen eines Kreiskegels auf zwei Arten in Kugelkoordinaten.

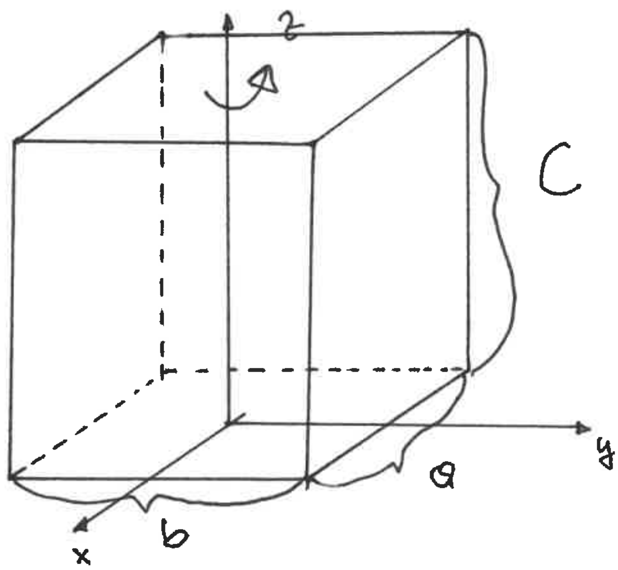
a.)



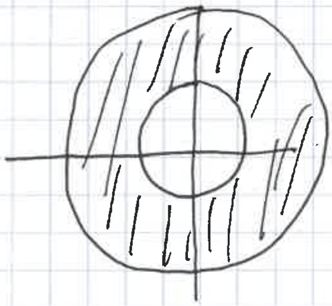
b.)



3.)
Berechnen sie das Trägheitsmoment eines Quaders mit konstanter Dichte wie er unten skizziert ist.



1a.)



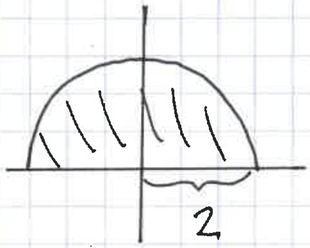
$$\int_{r=1}^{2}\int_{\varphi=0}^{2\pi} e^{-r^2}\, r\, dr\, d\varphi$$

$$= 2\pi \cdot \left(-\frac{1}{2}\right) e^{-r^2}\Big|_1^2 = -\pi \cdot \left(e^{-4} - e^{-1}\right)$$

$$= \underline{\underline{\pi \cdot \left(\frac{1}{e} - \frac{1}{e^4}\right)}}$$

1.b.)

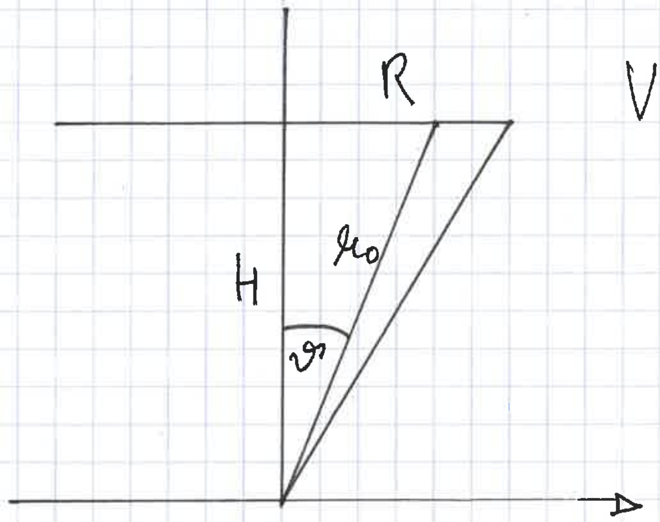
$$\int \int (x^2+y^2)^2 \, dx\, dy$$



$$\int_{r=0}^{2}\int_{\varphi=0}^{\pi} r^4 \, r\, dr\, d\varphi = \pi \cdot \frac{r^6}{6}\Big|_0^2$$

$$= \frac{\pi}{6} \cdot 64 = \underline{\underline{\frac{32}{3}\pi}}$$

2a)



$$V_{Kegel} = \frac{1}{3} R^2 \cdot \pi \cdot H$$

$$V = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\arctan\left(\frac{R}{H}\right)} \int_{r=0}^{\frac{H}{\cos\vartheta}} r^2 \sin\vartheta \, dr \, d\varphi \, d\vartheta$$

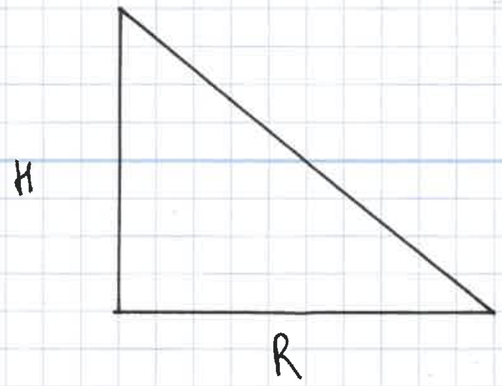
$$V = 2\pi \cdot \int_{\vartheta=0}^{\arctan\left(\frac{R}{H}\right)} \int_{r=0}^{\frac{H}{\cos\vartheta}} \sin\vartheta \, r^2 \, dr \, d\vartheta$$

$$V = 2\pi \int_{\vartheta=0}^{\arctan\left(\frac{R}{H}\right)} \sin\vartheta \cdot \frac{1}{3} r^3 \Big|_0^{\frac{H}{\cos\vartheta}} d\vartheta$$

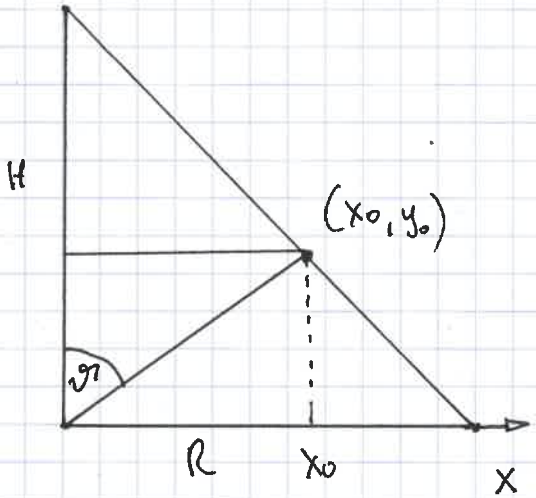
$$V = \frac{2}{3} \pi H^3 \int_{\vartheta=0}^{\arctan\left(\frac{R}{H}\right)} \frac{\sin\vartheta}{\cos^3\vartheta} \, d\vartheta = \frac{2}{3} \pi H^3 \cdot \int_{\vartheta=0}^{\arctan\left(\frac{R}{H}\right)} \frac{1}{\cos^2\vartheta} \cdot \tan(\vartheta) \, d\vartheta$$

$$V = \frac{\cancel{2}}{3} \pi H^3 \cdot \frac{1}{\cancel{2}} \tan^2(\vartheta) \Big|_0^{\arctan\left(\frac{R}{H}\right)} = \frac{1}{3} \pi H^3 \cdot \frac{R^2}{H^2} = \underline{\underline{\frac{1}{3} R^2 \pi \cdot H}}$$

2.b.



$$V = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\frac{\pi}{2}} \int_{r=0}^{?} r^2 \sin\vartheta \, dr \, d\varphi \, d\vartheta$$



$$r = \sqrt{x_0^2 + y_0^2}.$$

$$y = H - \frac{H}{R} \cdot x$$

$$y = \frac{r\cos\vartheta}{r\sin\vartheta} \cdot x = \cot\vartheta \cdot x$$

$$H - \frac{H}{R} \cdot x = \frac{\cos\vartheta}{\sin\vartheta} \cdot x$$

$$x \cdot \left(\frac{\cos\vartheta}{\sin\vartheta} + \frac{H}{R} \right) = H$$

$$x \cdot \frac{R\cos\vartheta + H\sin\vartheta}{R\sin\vartheta} = H$$

$$\underline{\underline{x_0 = \frac{RH\sin\vartheta}{R\cos\vartheta + H\sin\vartheta}}}$$

$$y = H - \frac{H}{\cancel{R}} \cdot \frac{\cancel{R}H\sin\vartheta}{R\cos\vartheta + H\sin\vartheta} = \frac{HR\cos\vartheta + \cancel{H^2\sin\vartheta} - \cancel{H^2\sin\vartheta}}{R\cos\vartheta + H\sin\vartheta} =$$

$$\underline{\underline{y_0 = \frac{R \cdot H \cdot \cos\vartheta}{R\cos\vartheta + H\sin\vartheta}}}$$

(II)

$$r = \sqrt{x_0^2 + y_0^2} = \sqrt{\frac{R^2H^2}{(R\cos\vartheta + H\sin\vartheta)^2}} = \frac{R \cdot H}{R\cos\vartheta + H\sin\vartheta}$$

$$V = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\frac{\pi}{2}} \int_{r=0}^{\frac{RH}{R\cos\vartheta + H\sin\vartheta}} r^2 \sin\vartheta \cdot dr \, d\varphi \, d\vartheta$$

$$V = \frac{2\pi}{3} \cdot \int_{\vartheta=0}^{\pi/2} r^3 \Big|_0^{\frac{RH}{R\cos\vartheta + H\sin\vartheta}} \sin\vartheta \, d\vartheta = \frac{2\pi}{3} R^3H^3 \int_{\vartheta=0}^{\pi/2} \frac{\sin\vartheta}{(R\cos\vartheta + H\sin\vartheta)^3} \, d\vartheta$$

$$= \frac{2\pi R^3H^3}{3} \cdot \int_{\vartheta=0}^{\pi/2} \frac{\sin(\vartheta)}{(\sin\vartheta \cdot (H + R\cot\vartheta))^3} \, d\vartheta = \frac{2\pi R^3H^3}{3} \cdot \int \frac{1}{\sin^2\vartheta} \cdot \frac{1}{(H + R\cot(\vartheta))^3} \, d\vartheta$$

$u = \cot\vartheta$

$du = -\frac{1}{\sin^2\vartheta} \, d\vartheta$

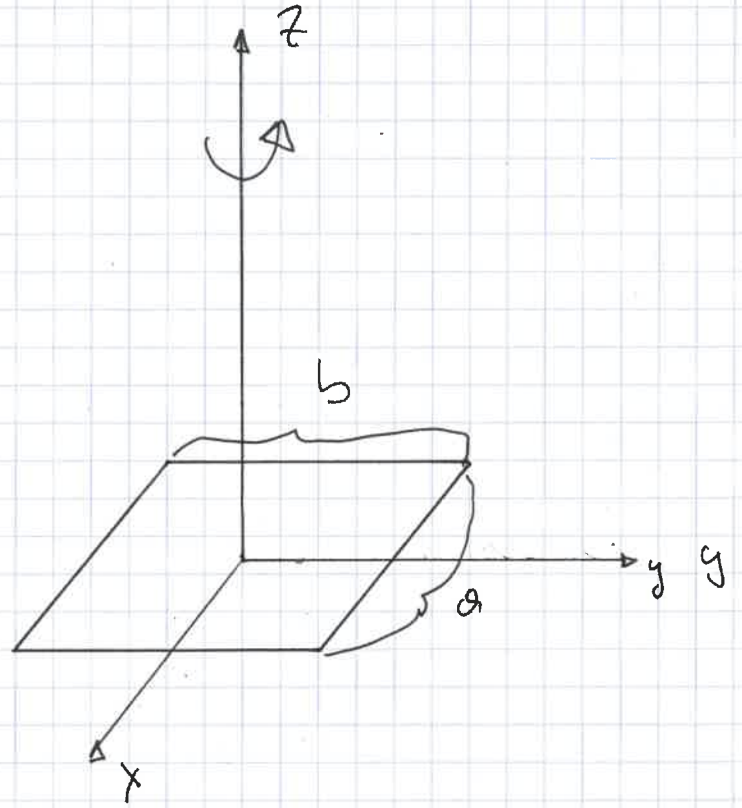
$$= \frac{2\pi R^3H^3}{3} \cdot (-1) \cdot \int \frac{du}{(H + R \cdot u)^3}$$

III

$$= \frac{2\pi R^3 H^3}{3} \cdot (-1) \cdot \int (H + R \cdot u)^{-3} \cdot du = \frac{\cancel{2}\pi R^3 H^3}{3} \cdot (+1) \cdot \frac{(H + R \cdot u)^{-2}}{+\cancel{2}} \cdot \frac{1}{R}$$

$$= \frac{1}{3} \pi R^3 H^3 \cdot \frac{1}{R} \cdot \frac{1}{(H + R \cot \vartheta)^2} \Bigg|_0^{\frac{\pi}{2}} = \frac{1}{3} \pi R^2 H^3 \cdot \left(\frac{1}{H^2} - 0 \right) = \underline{\underline{\frac{1}{3} R^2 \cdot \pi \cdot H}}$$

3.)



$$I = \int r^2 \cdot \rho \cdot dV$$

r: Normalabstand von der Drehachse eines jeden Punktes des Körpers.

$$I = \int_{x=-\frac{a}{2}}^{+\frac{a}{2}} \int_{y=-\frac{b}{2}}^{+\frac{b}{2}} \int_{z=0}^{c} (x^2+y^2) \cdot \rho \cdot dx\, dy\, dz$$

$$I = \rho \cdot c \int_{x=-\frac{a}{2}}^{+\frac{a}{2}} x^2 y + \frac{y^3}{3} \Big|_{-\frac{b}{2}}^{+\frac{b}{2}} dx$$

$$I = \rho \cdot c \cdot \int_{x=-\frac{a}{2}}^{+\frac{a}{2}} x^2 \left(\frac{b}{2} - \left(-\frac{b}{2}\right)\right) + \frac{1}{3} \cdot \left(\frac{b^3}{8} + \frac{b^3}{8}\right) dx$$

$$I = \rho \cdot c \cdot \int_{x=-\frac{a}{2}}^{+\frac{a}{2}} b \cdot x^2 + \frac{b^3}{12} \, dx$$

$$I = \rho \cdot c \cdot \left(b \, \frac{x^3}{3} + \frac{b^3}{12} \cdot x \right) \Big|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$I = \rho \cdot c \cdot \left(\frac{b}{3} \cdot \left(\frac{a^3}{8} + \frac{a^3}{8} \right) + \frac{b^3}{12} \cdot a \right)$$

$$I = \rho \cdot c \cdot \left(\frac{1}{12} \, b \cdot a^3 + \frac{1}{12} \, a b^3 \right)$$

$$I = \underbrace{\rho \cdot \underbrace{a \cdot b \cdot c}_{V}}_{M} \cdot \left(\frac{1}{12} \, a^2 + \frac{1}{12} \, b^2 \right)$$

$$\underline{\underline{I = \frac{1}{12} M \cdot (a^2 + b^2)}}$$