Übungen Mehrdimensionale Analysis am 09. 03. 2024

1.)

Berechnen sie die Winkel unter denen die Funktion $F(x,y) = 2x^3 + 6y^3 - 24y + 6x = 0$ die y - Achse schneidet.

2.)

Bestimmen sie die relativen Extremwerte der folgenden Funktionen.

a.) $z = 3xy^2 + 4x^3 - 3y^2 - 12x^2 + 1$

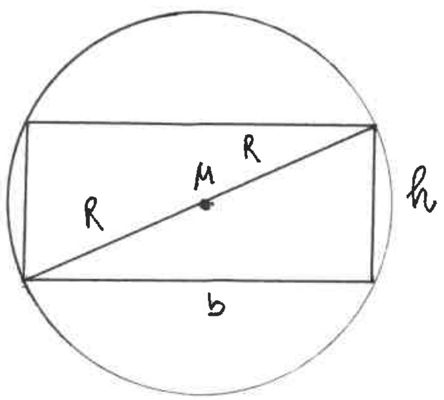
b.) $z = \sqrt{1 + x^2 + y^2}$

3.)

Bestimmen sie die Extremwerte der Funktion $z = x + y$ unter der Nebenbedingung $x^2 + y^2 = 1$.

4.)

Aus einem kreisrunden Baumstamm soll ein Balken mit rechteckigen Querschnitt so herausgeschnitten werden, dass sei Widerstandsmoment $W = \frac{bh^2}{6}$ maximal wird.



5.)

Welcher Punkt der Ebene $2x + 3y + z = 14$ hat vom Koordinatenursprung den kleinsten Abstand.

a.) $P_1 = (0,0)$

$$y' = -\frac{1}{-4} = \frac{1}{4} = \tan(\alpha_1)$$

$$\alpha_1 = \arctan\left(\frac{1}{4}\right) = \underline{\underline{14{,}04^\circ}}$$

Winkel mit x-Achse: $\underline{\underline{14{,}04^\circ}}$ mit y-Achse: $\underline{\underline{75{,}96^\circ}}$

b.) $P_2 = (0,2)$

$$y' = -\frac{1}{8}$$

$$\alpha_2 = \arctan\left(-\frac{1}{8}\right) = \underline{\underline{-7{,}13^\circ}}$$

Winkel mit y-Achse: $\underline{\underline{97{,}13^\circ}}$

c.) $P_3 = (0,-2)$

$$y' = \frac{-1}{8} \qquad \underline{\underline{\alpha_3 = -7{,}13^\circ}}$$

Winkel mit y-Achse : $\underline{\underline{97{,}13^\circ}}$

2a) $z = 3xy^2 + 4x^3 - 3y^2 - 12x^2 + 1$

$z_x = 3y^2 + 12x^2 - 24x = 0$ I

$z_y = 6xy - 6y = 0$ II

$z_{xx} = 24x - 24$

$\Delta = z_{xx} \cdot z_{yy} - z_{xy}^2$

$z_{yy} = 6x - 6$

$z_{xy} = 6y$

aus II: $6xy - 6y = 0$

$y \cdot (x-1) = 0 \Rightarrow \underline{\underline{x = 1}}, \underline{\underline{y = 0}}$

$x=1$: $3y^2 - 12 = 0$

$y^2 = 4$ $\underline{\underline{y_1 = 2}}$ $\underline{\underline{y_2 = -2}}$

$\underline{\underline{P_1 = (1,2)}}$ $\underline{\underline{P_2 = (1,-2)}}$

$y=0$: $12x^2 - 24x = 0$

$x^2 - 2x = 0$

$x \cdot (x-2) = 0$

$x_1 = 0$ $x_2 = 2$

$\underline{\underline{P_3 = (0,0)}}$

$\underline{\underline{P_4 = (2,0)}}$

$P_1 = (1,2)$

$\Delta = z_{xx} \cdot z_{yy} - z_{xy}^2 = 0 - 144 = -144 < 0$

kein Extremwert.

$P_2 = (1,-2)$

$\Delta = z_{xx} z_{yy} - z_{xy}^2 = -144 < 0$

kein Extremwert

$P_3 = (0,0)$

$\Delta = z_{xx} z_{yy} - z_{xy}^2 = -24 \cdot (-6) - (0)^2 = 144 > 0$

$z_{xx} = -24 < 0 \rightarrow$ Maximum

$P_4 = (2,0)$ $\quad \Delta = z_{xx} z_{yy} - z_{xy}^2 = 24 \cdot 6 = 144 > 0$

$z_{xx} = 24 > 0 \rightarrow$ Minimum

2.b) $z = \sqrt{1+x^2+y^2}$

$$z_x = \frac{1}{2} \cdot \frac{1}{\sqrt{\quad}} \cdot 2x = \frac{x}{\sqrt{1+x^2+y^2}} = 0$$

$$z_y = \frac{y}{\sqrt{1+x^2+y^2}} = 0$$

$$P = (0,0).$$

$$z_{xx} = \frac{\sqrt{\quad} - x \cdot \cancel{\frac{1}{2}} \frac{1}{\sqrt{\quad}} \cdot \cancel{2}x}{(1+x^2+y^2)} = \frac{1+x^2+y^2-x^2}{(1+x^2+y^2)^{3/2}} = \frac{1+y^2}{(1+x^2+y^2)^{3/2}}$$

$$z_{yy} = \frac{1+x^2}{(1+x^2+y^2)^{3/2}}$$

$$z_{xy} = \frac{-x \cdot \cancel{\frac{1}{2}} \frac{1}{\sqrt{\quad}} \cdot \cancel{2}y}{(1+x^2+y^2)} = \frac{-xy}{(1+x^2+y^2)^{3/2}}$$

$$\Delta = z_{xx} \cdot z_{yy} - z_{xy}^2 = 1 \cdot 1 - 0 = 1 > 0$$

$z_{xx} = 1 \rightarrow$ Minimum a $P = (0,0)$

3) $z = x+y \qquad x^2+y^2 = 1$

$F(x,y) = z + \lambda \cdot (x^2+y^2-1)$

$F(x,y) = x+y + \lambda(x^2+y^2-1)$

$\left.\begin{array}{l} F_x = 1 + 2x\lambda = 0 \qquad \lambda = -\frac{1}{2x} \\ F_y = 1 + 2y\lambda = 0 \qquad \lambda = -\frac{1}{2y} \end{array}\right\} \quad \underline{\underline{x = y}}$

$F_\lambda = x^2+y^2-1 = 0$

$2x^2 = 1$

$x = \pm\frac{1}{\sqrt{2}} = \underline{\underline{\pm\frac{1}{2}\sqrt{2}}} \Rightarrow \underline{\underline{y = \pm\frac{1}{\sqrt{2}} = \pm\frac{1}{2}\sqrt{2}}}$

$P_1 = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \qquad P_2 = \left(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right)$

$z(P_1) = \sqrt{2} \qquad z(P_2) = -\sqrt{2}$

$\Downarrow \qquad \Downarrow$

Maxim   Minim.

4.) $W = \frac{bh^2}{6}$ $\quad b^2 + h^2 = 4R^2$

$h^2 = 4R^2 - b^2$

1. Weg:

$$W = \frac{1}{6} \quad b \cdot \left(4R^2 - b^2\right) = \frac{1}{6} \left(4R^2 b - b^3\right)$$

$$\frac{dW}{db} = \frac{1}{6} \cdot \left(4R^2 - 3b^2\right) \stackrel{!}{=} 0$$

$$3b^2 = 4R^2$$

$$\underline{\underline{b = \frac{2}{\sqrt{3}} R}} \qquad h = \sqrt{4R^2 - \frac{4}{3}R^2} = R \cdot \sqrt{\frac{8}{3}} = \underline{\underline{\frac{2 \cdot \sqrt{2}}{\sqrt{3}} R}}$$

$$\frac{d^2W}{db^2} = -6b < 0 \Rightarrow \text{Maximum}$$

$\underline{\underline{b = 1{,}155\ R}}$ $\qquad$ $\underline{\underline{h = 1{,}633\ R}}$

$$\underline{\underline{W_{max} = 0{,}513\ R^3}}$$

4.) $W = \frac{1}{6} b \cdot h^2$ $\qquad b^2 + h^2 = 4R^2$

2. Weg:

$$F(b, h, \lambda) = \frac{1}{6} b h^2 + \lambda \cdot \left(b^2 + h^2 - 4R^2 \right)$$

$$\mathrm{I}: F_b = \frac{1}{6} h^2 + 2b\lambda = 0$$

$$\mathrm{II}: F_h = \frac{1}{3} b \cdot h + 2h\lambda = 0 \quad \rightarrow \underline{\underline{\lambda = -\frac{b}{6}}}$$

$$\mathrm{III}: F_\lambda = b^2 + h^2 - 4R^2 = 0$$

Einsetzen in I: $\frac{1}{6} h^2 + 2b \cdot \left(-\frac{b}{6} \right) = 0$

$$h^2 - 2b^2 = 0$$

$$\underline{\underline{h^2 = 2b^2}} \qquad \underline{\underline{h = b \cdot \sqrt{2}}}$$

in III: $b^2 + 2b^2 = 4R^2$

$$b^2 = \frac{4}{3} R^2$$

$$\underline{\underline{b = \frac{2}{\sqrt{3}} R}} \qquad \underline{\underline{h = 2 \cdot \frac{\sqrt{2}}{\sqrt{3}} R}}$$ wie unter 4.) 1. Weg

5.) $\mathcal{E}: \quad 2x + 3y + z = 14$

Abstand: $d = \sqrt{x^2+y^2+z^2} \rightarrow$ Minimalwert.

$$F(x,y,z) = \sqrt{x^2+y^2+z^2} + \lambda \cdot (2x+3y+z-14)$$

$$F_x = \frac{x}{\sqrt{x^2+y^2+z^2}} + 2\lambda = 0 \qquad \text{I}$$

$$F_y = \frac{y}{\sqrt{x^2+y^2+z^2}} + 3\lambda = 0 \qquad \text{II}$$

$$F_z = \frac{z}{\sqrt{x^2+y^2+z^2}} + \lambda = 0 \quad \text{III} \rightarrow \lambda = -\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$F_\lambda = 2x+3y+z-14 = 0 \qquad \text{IV}$$

Einsetzen in I: $\frac{x}{\sqrt{x^2+y^2+z^2}} - \frac{2z}{\sqrt{x^2+y^2+z^2}} = 0 \qquad \underline{\underline{x = 2z}}$

Einsetzen in II: $\frac{y}{\sqrt{x^2+y^2+z^2}} - \frac{3z}{\sqrt{x^2+y^2+z^2}} = 0 \qquad \underline{\underline{y = 3z}}$

Einsetzen in IV: $4z + 9z + z = 14$

$\underline{\underline{z=1}} \quad \underline{\underline{x=2}} \quad \underline{\underline{y=3}}$

Der Punkt $P = (2,3,1)$ hat minimalen Abstand $\underline{\underline{d = \sqrt{14}}}$

1.) $F(x,y) = 2x^3 + 6y^3 - 24y + 6x = 0.$

Schnittpunkt mit der y-Achse. $x = 0.$

$$6y^3 - 24y = 0$$

$$y^3 - 4y = 0$$

$$y \cdot (y^2 - 4) = 0$$

$y_1 = 0$ $y_2 = 2$ $y_3 = -2$

$P_1 = (0,0)$ $P_2 = (0,2)$ $P_3 = (0,-2)$

$$F(x,y) = 2x^3 + 6y^3 - 24y + 6x = 0 \quad \Big| \frac{d}{dx}$$

$$6x^2 + 18y^2 y' - 24y' + 6 = 0$$

$$y' \cdot (18y^2 - 24) = \frac{-6x^2 - 6}{18y^2 - 24}$$

$$\underline{\underline{y' = -\frac{x^2 + 1}{3y^2 - 4}}}$$

4.)

$$W = \frac{b \cdot h^2}{6} \qquad b^2 + h^2 = 4R^2$$

1. Weg:

$$h^2 = 4R^2 - b^2$$

$$W = \frac{1}{6} \quad b \cdot (4R^2 - b^2)$$

$$W = \frac{1}{6} \cdot (4R^2 b - b^3)$$

$$\frac{dW}{db} = \frac{1}{6} \cdot (4R^2 - 3b^2) \overset{!}{=} 0$$

$$3b^2 = 4R^2$$

$$b^2 = \frac{1}{3} 4R^2$$

$$b = \frac{1}{\sqrt{3}} \cdot 2 \cdot R = \frac{2}{3} \cdot \sqrt{3} \cdot R \qquad h = \sqrt{4R^2 - \frac{4}{3}R^2}$$

$$h = R \cdot \frac{2}{\sqrt{3}} \cdot \sqrt{2}$$

$$\frac{d^2W}{db^2} = -6b$$

$$\frac{d^2W}{db^2}\left(b = \frac{2}{3}\sqrt{3}R\right) < 0$$

$$W =$$