Übungen Mathematik II am 11. 05. 2024

1.)

Berechnen sie die Tangenten und Hauptnormaleneinheitsvektoren und die Krümmung folgender Kurve für den Parameterwert t = $\frac{\pi}{4}$.

$$\vec{r} = \begin{pmatrix} 2\cos(5t) \\ 2\sin(5t) \\ 10t \end{pmatrix}$$

2.)

Berechnen sie die Bogenlänge Zwischen den Punkten t = 0 und t = 1.

$$\vec{r} = \begin{pmatrix} t \\ t^2 \\ t^2 + t \end{pmatrix}$$

3.)

Gegeben sind die folgenden parameterabhängigen Vektoren.

$$\vec{a} = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2.\cos(t) \\ 2.\sin(t) \\ t^2 \end{pmatrix}$$

Berechnen sie die ersten Ableitungen von $\vec{a}\ x\ \vec{b}$ und $\vec{a}.\vec{b}$.

4.)

Bestimmen sie für die Raumkurve $\vec{r} = \begin{pmatrix} t^2 \\ t \\ t^2 \end{pmatrix}$ folgende Grüßen:

a.) Bogenlänge zwischen t = 0 und t = 1.

b.) Krümmung und Krümmungsradius für t = 1.

1.)

$$\vec{r} = \begin{pmatrix} 2\cos(5t) \\ 2\sin(5t) \\ 10t \end{pmatrix}$$

$$\dot{\vec{r}} = \begin{pmatrix} -10\sin(5t) \\ 10\cos(5t) \\ 10 \end{pmatrix} \qquad \ddot{\vec{r}} = \begin{pmatrix} -50\cos(5t) \\ -50\sin(5t) \\ 0 \end{pmatrix}$$

$$T = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \begin{pmatrix} -10\sin(5t) \\ 10\cos(5t) \\ 10 \end{pmatrix} \cdot \frac{1}{\sqrt{200}} = \underline{\underline{\begin{pmatrix} -\frac{1}{\sqrt{2}}\sin(5t) \\ \frac{1}{\sqrt{2}}\cos(5t) \\ \frac{1}{\sqrt{2}} \end{pmatrix}}}$$

$$T\left(t = \frac{\pi}{4}\right) = \underline{\underline{\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}}}$$

$$\vec{N} = \frac{\dot{\vec{T}}}{|\dot{T}|} = \begin{pmatrix} -\frac{5}{\sqrt{2}}\cos(5t) \\ -\frac{5}{\sqrt{2}}\sin(5t \\ 0 \end{pmatrix} \cdot \frac{1}{\frac{5}{\sqrt{2}}} = \underline{\underline{\begin{pmatrix} -\cos(5t) \\ -\sin(5t) \\ 0 \end{pmatrix}}}$$

$$\underline{\underline{\vec{N}\left(t = \frac{\pi}{4}\right) = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix}}}$$

$$\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{pmatrix} -10 \sin(5t) \\ 10 \cos(5t) \\ 10 \end{pmatrix} \times \begin{pmatrix} -50 \cos(5t) \\ -50 \sin(5t) \\ 0 \end{pmatrix} = \begin{pmatrix} 500 \sin(5t) \\ -500 \cos(5t) \\ 500 \end{pmatrix}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \left(500^2 + 500^2\right)^{1/2} = \underline{\underline{500 \cdot \sqrt{2}}}$$

$$|\dot{\vec{r}}| = \left(100 + 100\right)^{1/2} = 10 \cdot \sqrt{2}$$

$$\kappa = \frac{500 \cdot \sqrt{2}}{1000 \cdot 2 \cdot \sqrt{2}} = \underline{\underline{\frac{1}{4}}}$$

immer konstant

auch bei $t = \frac{\pi}{4}$.

2.) $\vec{r} = \begin{pmatrix} t \\ t^2 \\ t^2+t \end{pmatrix}$ $\dot{r} = \begin{pmatrix} 1 \\ 2t \\ 2t+1 \end{pmatrix}$

$$S = \int_0^1 \sqrt{1+4t^2+4t^2+4t+1}\,dt = \int_0^1 \sqrt{2+8t^2+4t}\,dt = \int_0^1 \sqrt{2+8\cdot\left(t^2+\frac{t}{2}\right)}\,dt =$$

$$= \int_0^1 \sqrt{2+8\cdot\left(\left(t+\frac{1}{4}\right)^2-\frac{1}{16}\right)}\,dt = \int_0^1 \sqrt{\frac{3}{2}+8\cdot\left(t+\frac{1}{4}\right)^2} = \int_0^1 \sqrt{\frac{3}{2}}\cdot\sqrt{1+\frac{16}{3}\cdot\left(t+\frac{1}{4}\right)^2}\,dt$$

$$= \sqrt{\frac{3}{2}}\cdot\int_0^1 \sqrt{1+\left[\frac{4}{\sqrt{3}}\left(t+\frac{1}{4}\right)\right]^2}\,dt$$

Subst: $\frac{4}{\sqrt{3}}\cdot\left(t+\frac{1}{4}\right) = \sinh(u)$

$\frac{4}{\sqrt{3}}\,dt = \cosh(u)\,du$

$dt = \frac{\sqrt{3}}{4}\cosh(u)\,du$

$$\frac{3}{4\cdot\sqrt{2}}\left(\int \frac{1}{2}\left(1+\cosh(2u)\right)du = \frac{3}{8\cdot\sqrt{2}}\cdot\left(u+\frac{1}{2}\sinh(2u)\right)\right.$$

$$= \frac{3}{8\cdot\sqrt{2}}\cdot\left(u+\sinh(u)\cosh(u)\right) = \frac{3}{8\cdot\sqrt{2}}\cdot\left(\operatorname{arcsinh}\left(\frac{4}{\sqrt{3}}\cdot\left(t+\frac{1}{4}\right)\right)+\frac{4}{\sqrt{3}}\left(t+\frac{1}{4}\right)\cdot\sqrt{1+\frac{16}{3}\left(t+\frac{1}{4}\right)^2}\right)\Bigg|_0^1$$

$$= \frac{3}{8\cdot\sqrt{2}}\left(\operatorname{arcsinh}\left(\frac{4}{\sqrt{3}}\cdot\frac{5}{4}\right)-\operatorname{arcsinh}\left(\frac{1}{\sqrt{3}}\right)+\frac{4}{\sqrt{3}}\cdot\left(\frac{5}{4}\right)\cdot\sqrt{1+\frac{16}{3}\cdot\left(\frac{25}{16}\right)}-\frac{4}{\sqrt{3}}\cdot\frac{1}{4}\cdot\sqrt{1+\frac{1}{3}}\right) = 2,48863$$

3.) $\vec{a} = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ $\qquad$ $\vec{b} = \begin{pmatrix} 2\cos(t) \\ 2\sin(t) \\ t^2 \end{pmatrix}$

$$\vec{a} \cdot \vec{b} = 2t\cos(t) + 2t^2\sin(t) + t^5$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} t^4 - 2t^3\sin(t) \\ 2t^3\cos(t) - t^3 \\ 2t\sin(t) - 2t^2\cos(t) \end{pmatrix}$$

$$\frac{d}{dt}(\vec{a} \cdot \vec{b}) = \underline{\underline{2\cos(t) - 2t\sin(t) + 4t\sin(t) + 2t^2\cos(t) + 5t^4}}$$

$$= \underline{\underline{2t\sin(t) + 2\cos(t) + 2t^2\cos(t) + 5t^4}}$$

$$\frac{d}{dt}(\vec{a} \times \vec{b}) = \begin{pmatrix} 4t^3 - 6t^2\sin(t) - 2t^3\cos(t) \\ 6t^2\cos(t) - 2t^3\sin(t) - 3t^2 \\ 2\sin(t) + 2t\cos(t) - 4t\cos(t) + 2t^2\sin(t) \end{pmatrix}$$

4a)

$$S = \int_{t1}^{t2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \cdot dt$$

$$S = \int_0^1 \sqrt{(2t)^2 + 1 + (2t)^2}\, dt = \int_0^1 \sqrt{1 + 8t^2}\, dt$$

$$\cosh^2(t) = 1 + \sinh^2(t)$$

$$\sqrt{8} \cdot t = \sinh(t)$$

$$dt = \frac{1}{\sqrt{8}} \cosh(t)\, dt$$

$$S = \int \underbrace{\sqrt{1 + \sinh^2(t)}}_{\cosh(t)} \cdot \frac{1}{\sqrt{8}} \cosh(t)\, dt = \frac{1}{\sqrt{8}} \cdot \int \cosh^2(t)\, dt$$

$$\cosh^2(t) - \sinh^2(t) = 1$$

$$\cosh^2(t) + \sinh^2(t) = \cosh(2t)$$

$$\cosh^2(t) = \frac{1}{2} \cdot (1 + \cosh(2t))$$

$$S = \frac{1}{\sqrt{8}} \cdot \frac{1}{2} \int 1 + \cosh(2t)\, dt = \frac{1}{2 \cdot \sqrt{8}} \cdot \left(t + \frac{1}{2}\sinh(2t)\right)$$

$$S = \frac{1}{2 \cdot \sqrt{8}} \cdot \left(\operatorname{arsinh}\left(\sqrt{8} \cdot t\right) + \cancel{\tfrac{1}{2}} \cdot \cancel{2}\sinh(t)\cosh(t) \right)$$

$$S = \frac{1}{2\sqrt{8}} \cdot \left(\operatorname{arsinh}(\sqrt{8} \cdot t) + \sqrt{8} \cdot t \cdot \sqrt{1+8t^2} \right) \Big|_0^1$$

$$= S = \frac{1}{2\sqrt{8}} \left(\operatorname{arsinh}(\sqrt{8}) + \sqrt{8} \cdot \sqrt{1+8} \right)$$

$$S = \frac{1}{2 \cdot \sqrt{8}} \cdot \left(\ln\left(\sqrt{8} + \sqrt{1+8} \right) + \sqrt{8} \cdot \sqrt{1+8} \right)$$

$$S = \frac{1}{2\sqrt{8}} \cdot \left(\ln\left(\sqrt{8} + 3 \right) + \sqrt{8} \cdot 3 \right)$$

$$\underline{\underline{S = 1{,}8116}}$$

4.b) $\varkappa = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$

$$\dot{\vec{r}} = \begin{pmatrix} 2t \\ 1 \\ 2t \end{pmatrix} \qquad \ddot{\vec{r}} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$|\dot{\vec{r}}| = \underline{\underline{\sqrt{1+8t^2}}}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{pmatrix} 2t \\ 1 \\ 2t \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}}}$$

$$\underline{\underline{\varkappa = \frac{\sqrt{8}}{(1+8t)^{3/2}}}} \qquad \varkappa(t=1) = \frac{\sqrt{8}}{27} = \underline{\underline{0{,}10475}}$$

$$\rho(t=1) = 9{,}55$$